

Entanglement subspaces, trial wavefunctions, and special Hamiltonians in the fractional quantum Hall effect

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(Dated: March 19, 2013)

We consider systems of trial wavefunctions in the fractional quantum Hall effect that can be obtained in various ways. In one way, the functions are obtained by analyzing the entanglement of the ground state wavefunction, partitioned into two parts. In another, functions are defined by the way in which they vanish as several coordinates approach the same value, or by a projection operator Hamiltonian that enforces those conditions. In a third way, the functions are given by conformal blocks from a conformal field theory (CFT). These different sets are closely related. The use of CFT methods permits an algebraic formulation to be given for all of them. In some cases, we can prove that all of these spaces are the same (a finite-size bulk-edge correspondence), thus answering several questions and conjectures. We can also use the analysis of functions to produce a projection operator Hamiltonian that produces them as zero-energy states. For a model related to the $\mathcal{N} = 1$ superconformal algebra, the corresponding Hamiltonian imposes vanishing properties involving only three particles; for this we determine all the wavefunctions explicitly. We do the same for a sequence of models involving the $M(3, p)$ Virasoro minimal models that has been considered previously, using results from the literature. We exhibit the Hamiltonians for the first few cases of these.

I. INTRODUCTION

A. Overview

In recent years there has been a considerable growth of interest in topological phases of matter (see e.g. Ref. [1, 2] for reviews). A topological phase of matter can be defined as an equilibrium zero-temperature phase of a quantum system that has a gap in its energy spectrum above the ground state, as far as local excitations far from any boundary are concerned. Topological phases may have non-trivial properties (also called topological) that are unchanged by any (local) perturbation of the (local) Hamiltonian unless it causes the system to pass through a phase transition. Model examples of systems (consisting of a space of states, and a Hamiltonian) lying in a particular topological phase provide existence proofs for the phase. Due to the gap in the spectrum, any sufficiently small perturbation of the Hamiltonian by a local term will not destroy the gap. Hence, the solutions (even partial solutions) of such model examples provides paradigms for those phases. Among the earliest examples of topological phases and of the use of partially soluble models is the fractional quantum Hall effect (FQHE) [3, 4].

Historically, a standard approach has been the use of trial (or variational) states for the ground and some of the excited states, even in the absence of a soluble Hamiltonian. In the FQHE, examples of such states are usually described by a wavefunction that may readily be written down for arbitrary size (particle number) in terms of multiplying some factors, performing integrals or sums, or other operations. It is then hoped that these trial or “nice” wavefunctions have the topological properties of

a topological phase. In some cases, a Hamiltonian (or rather, a family of Hamiltonians, one for each system size) is known for which the ground state is also known exactly for each size of the system. As generic states are hard to specify concisely, such a state must have some special structure that enables it to be specified in a simple way; thus it too must be a nice function.

Various approaches to obtaining trial wavefunctions have been used in the past. In the FQHE, after Laughlin’s wavefunctions [3], there were generalizations using the hierarchy and composite particle approaches [4–6], though some of these schemes possess considerable leeway when it comes to specifying definite functions. In this paper we will be concerned with another approach that developed from Laughlin’s states. It stemmed from the observation that the algebraic behavior of the functions, which vanish as a power of separation as any two particles approach the same point, implies that there is a “pseudopotential” interaction Hamiltonian acting in the space of lowest-Landau level (LLL) states that annihilates both the Laughlin ground state and also the states containing any number of fractionally-charged quasihole excitations [4]. Thus the “nice” wavefunctions are actually zero-energy eigenstates of a “special” Hamiltonian, and also exhaust all such states. The special Hamiltonian consists of projection operators acting within the LLL; the coefficients of the projectors are positive, so the Hamiltonian is positive (semidefinite) as an operator within the LLL. Moreover, the special Hamiltonian is usually translationally and rotationally invariant. It is also local (short range), because it involves the low-degree behavior of the wavefunction in the differences of particle coordinates as they tend to zero.

Another approach to obtain trial wavefunctions was

developed by Moore and Read (MR) [7], who observed that the Laughlin wavefunctions, and some other previously-known trial wavefunctions, are chiral correlation functions (so-called conformal blocks) in some two-dimensional conformal field theory (CFT) [8, 9]. They extended this observation to propose some further trial states, and the method used amounts to a technique for the construction of an unlimited number of trial ground and associated quasihole wavefunctions. Subsequently, it was observed [10, 11] that the main example in MR, the MR or Pfaffian state, is itself the zero-energy ground state of a special (i.e. projection operator) Hamiltonian. This Hamiltonian is similar to those for the Laughlin state, in that it involves the way in which the function vanishes as any *three* particle coordinates come to the same point, and so consists of a three-body interaction. Using this Hamiltonian, explicit wavefunctions for any number of quasiholes were constructed [11, 12]. Other examples with similar structure have been found, including notably a sequence of further distinct topological phases in which the special Hamiltonians contain $k + 1$ -body interactions, where $k = 3, 4, 5, \dots$ [13]; the preceding examples fit into the sequence as the cases $k = 1, 2$.

Of the sets of ground and quasihole trial LLL wavefunctions that have been proposed, many of those that can be expressed as functions, without use of integration over dummy variables (such as use of projection operators, possibly from a larger Hilbert space such as those including higher Landau levels), can be viewed as conformal blocks in a (known) CFT. Further, to our knowledge, *all* such sets of functions, that appear acceptable as a space of ground and quasihole states in a topological phase and are precisely the zero-energy states of a LLL special Hamiltonian, are conformal blocks. It should be emphasized that even when all the zero-energy states of a special Hamiltonian can be found, the non-zero energy eigenvalues generally cannot, and one does not know directly whether there is a gap in the spectrum of bulk excitations over the ground state in the infinite size limit. Indeed, it has been argued that when the wavefunctions are conformal blocks of either a non-rational or non-unitary CFT, any such Hamiltonian must be gapless in the limit [14, 15]. Consequently, in these cases the wavefunctions do not represent the behavior of a topological phase, but may be at a critical point.

In this paper, the main goal is to provide tools to study systematically the short-distance behavior of wavefunctions (and related entanglement properties), use this to find special Hamiltonians, and analyze the spaces of all zero-energy states of those Hamiltonians. The properties of the wavefunctions in these spaces lends itself to an algebraic interpretation in general, and leads almost inevitably to the use of MR constructions from a CFT to construct and analyze examples. In giving an overview of these ideas and results to be presented in this paper, we will now divide them under the following headings.

1. Zero-energy subspaces for a given Hamiltonian

LLL wavefunctions for bosons are essentially symmetric polynomials in the complex coordinates $z_j = x_j + iy_j$ for the j th particle with Cartesian coordinates x_j, y_j in the plane. From here on, when we refer to functions, it is such symmetric polynomials that we mean. The phrase “short-distance behavior” means a Taylor expansion in some selected particle coordinates as they all tend to some value; hence we usually organize the spaces by degree of the polynomials, beginning with the lowest degree. (We concentrate on bosons; wavefunctions for fermions may be obtained by multiplying by $\prod_{i < j} (z_i - z_j)$, and there are related Hamiltonians.) Given a translationally- and rotationally-invariant projection operator Hamiltonian that consists of interactions among at most N_{\max} particles, one can in principle determine all of the zero-energy states for each number of particles $N = 1, 2, \dots$ in the plane. We call the spaces of these states the *zero-energy (sub-)spaces* of states or wavefunctions; they are subspaces of the space of all LLL states for given N , and usually only of interest when they are *proper* subspaces. We also describe the wavefunctions as having “allowed” behavior (one might also describe them as “satisfying” the constraints that they are annihilated by the projectors in the special Hamiltonian). In the literature, these are often called spaces of edge excitations, though they can only be considered as true edge excitations in the case when there is a gap for bulk excitations [16, 17]. (One may also consider other geometries, including the sphere with a given number of flux quanta emerging from it, which lead to related results. For a gapped bulk phase on the sphere, the zero-energy states may be viewed as quasihole states.) In the body of this paper, we carry out the analysis of the zero-energy subspaces for the plane and the sphere explicitly in some examples.

2. Entanglement subspaces for a given ground state

In analyzing the zero-energy states, the spaces for $N \leq N_{\max}$ play a special role. If these are known, then the zero-energy wavefunctions for $N > N_{\max}$ particles must be found by some process of “patching together” the zero-energy wavefunctions for sets of $N_A \leq N_{\max}$ particles. Put another way, if the former are decomposed into wavefunctions for a subset of $N_A \leq N_{\max}$ particles and for the remainder, the wavefunctions for the N_A particles must lie in the zero-energy subspace. Indeed, the same is true for the decomposition for *any* value of $N_A \leq N$. For the “ground state” in the zero-energy subspace for any N , that is the (usually unique) wavefunction of lowest total degree in the z_i ’s (lowest angular momentum), we term the space of functions that actually occur when it is decomposed into functions of N_A , times functions of the remaining $N - N_A$, coordinates an *entanglement subspace*, because of its relation (discussed below) to entanglement properties of a ground state wavefunction.

From the preceding remarks, it is clear that the entanglement subspaces for any N_A (for the ground state at any $N \geq N_A$) must lie within the zero-energy subspace for $N = N_A$ particles. [One could of course do the same for any zero-energy state; we sometimes refer to these also as entanglement subspaces.] In the body of this paper, we analyze the relation of the zero-energy and entanglement subspaces, in some examples.

3. Inverse problem: finding a special Hamiltonian for given zero-energy subspaces

If there are some polynomials “missing” from the entanglement subspace for some value of N_A , and some degree d , for all values of the total number N , then a projection operator onto that function can be added to the Hamiltonian, and it will still annihilate the wavefunctions. More precisely, the projection must be onto some part of the orthogonal complement of the “allowed” functions. (The orthogonal complement and the construction of projection operators involve the use of the quantum-mechanical inner product on the wavefunctions of our system.) The special Hamiltonian with which we began must itself consist of such projectors, and one would like to use one with a minimal number of terms. Then we also have the inverse problem: if we are given some spaces of wavefunctions, for all particle numbers N , and we wish to obtain a special Hamiltonian for which these are precisely the zero-energy subspaces, then such an analysis of the entanglement subspaces will ultimately lead to the solution, if one exists. A necessary condition for existence is that the spaces of functions should obey the compatibility condition mentioned in the preceding paragraph. That is, all the entanglement subspaces for N_A particles in *any* N -particle function in the given space ($N_A < N$) should be a subspace of the given space of functions for N_A particles.

4. Spaces of amplitudes from a CFT

In particular, as spaces of functions, we can begin with conformal blocks in a MR construction. This produces spaces of functions, which we may also term “spaces of amplitudes”, that play the role of the zero-energy subspaces; the spaces depend on the CFT used. In these constructions, the necessary compatibility conditions are automatically satisfied, because the short-distance analysis of entanglement subspaces is related to the operator-product expansion in CFT, which has such behavior [13]. We may attempt to find a corresponding special Hamiltonian for which such spaces of amplitudes are the zero-energy subspaces.

5. Entanglement

The analysis so far has led to issues of entanglement. For entanglement in general in a many-particle state, one effectively divides the system into one part with N_A particles, and the remainder with $N_B = N - N_A$ particles. One may trace over the states of the N_B particles to obtain a reduced density matrix, and then analyze its eigenvectors and eigenvalues [18, 19]. Thus essentially, the space in which the eigenvectors lie (the entanglement subspace) is that spanned by the wavefunctions of the N_A particles that occur in the ground state, just as in the preceding discussion (we will discuss different ways of defining the bipartition of the system in more depth later). If the partition respects rotation invariance about the origin, then the states can be classified according to their angular momentum, which is the same as their degree as polynomials. In some ways of defining the bipartition to obtain entanglement, a range of values of N_A must be considered also. Several ways of bipartitioning for LLL states have been considered in the literature; for our purposes, which relate to special Hamiltonians, the most appropriate is particle partition [20], which preserves translational, as well as rotational, symmetry. (Many studies of entanglement use the sphere; in that geometry particle partition has the full rotation symmetry.) When studying entanglement, one is often interested in universal behavior as N_A (or its “typical” value, if there is a range) tends to infinity (either subsequent to, or at the same time as, $N \rightarrow \infty$).

An important idea is that in LLL trial states for a topological phase in the FQHE (in fact, for those obtained as conformal blocks/amplitudes from a MR construction), the multiplicities of the states in the decomposition at sufficiently large angular momentum and N_A —that is, the dimensions of the entanglement subspaces—approach those of the edge states [21], which can sometimes be identified as the zero energy states of a special Hamiltonian. For states given by the MR construction as conformal blocks, these multiplicities (in the limit) are those of the vacuum sector of the CFT. More generally, in that construction, the entanglement subspaces must lie within the spaces of amplitudes, because of the compatibility conditions already mentioned. We will show, by general algebraic arguments, that the dimension of the space of *amplitudes* obtained by the MR construction *does* agree with the vacuum sector of the CFT used as $N \rightarrow \infty$ (in an appropriate way). We also show that, in the limit $N_B \rightarrow \infty$ with N_A fixed, the entanglement subspaces contain *all* the same functions as those in the spaces of amplitudes for $N = N_A$, at *any finite* value of N_A as well, provided the correct CFT is used to construct amplitudes. Hence in particular, their dimensions are equal. This is a finite size (N_A finite, though N_B went to infinity) version of a bulk-edge correspondence, if the spaces of amplitudes are viewed as edge states. The “correct” CFT is one without any “singular” vectors, or alternatively can be obtained from the given CFT by a process

of setting the singular vectors to zero, to form the quotient space [8, 9]. (Singular vectors can only occur in non-unitary CFTs.) The equality of the *entanglement* subspaces with the spaces of amplitudes does *not* hold for general N_A as $N_B \rightarrow \infty$ if there are singular vectors in the CFT that was used to find the amplitudes, even though the ground state wavefunctions, that result from the construction using the CFT instead of its quotient, are the same. We give examples of this phenomenon.

6. Algebraic techniques

We develop an algebraic language to aid in all of this analysis. The space of all symmetric polynomials in N_A variables is a vector space, and can be viewed as the Hilbert space of LLL states of bosons. But it can also be viewed as an algebra, because not only are sums and complex multiples of symmetric polynomials again symmetric polynomials in the same variables, but so are products of them. Further, this algebra acts in an obvious way, by multiplication, on the vector space of symmetric polynomials, which can then also be viewed as a “module” over (or in less fancy language, a representation of) this algebra.

It is very convenient if the zero-energy subspaces, which are subspaces of symmetric polynomials, are themselves modules over the symmetric polynomials—that is, if the subspace is closed under multiplication by symmetric polynomials. That is because, as we will see when explicitly finding allowed functions, this property allows us to combine factors for different subsets of the particles that satisfy the conditions on allowed functions separately, secure in the knowledge that they will also do so when combined as N particle wavefunctions. This property holds in previously known examples, and in those we analyze here. If the special Hamiltonian consisted of an arbitrary set of projectors (i.e. if the allowed functions did not form a module over the symmetric polynomials), it would be much harder to construct the spaces of zero-energy many-particle states, and it would not be clear if any such states even exist. We further require that our Hamiltonians be short-range interactions, and that they be translationally and rotationally invariant. Rotational invariance implies that angular momentum, or the degree of the polynomials, be a good quantum number; we may classify functions by degree. Translational invariance implies that the allowed behavior is mapped into itself by the application of the center of mass variable $\propto \sum_i z_i$ and $\sum_i \partial/\partial z_i$ (acting on the polynomials).

Consequently, when searching for special Hamiltonians, we now have an algebraic problem, involving the study of translation-invariant modules for the algebra of symmetric polynomials. While this problem can be solved completely for the case of $N_A = 2$ particles (see Section IC below), it is much more complex for more than two particles. To make progress, motivated by the MR construction using CFT, we can also enlarge the al-

gebra of symmetric polynomials by introducing further operators which form part of the action of the Virasoro algebra on the wavefunctions. This algebraic problem can also be studied, without further use of CFT. Because the requirement of being a module over the larger algebra is more stringent, there are fewer solutions to be found, and they potentially arise from conformal blocks. Conversely, given such a Hamiltonian, we may find its zero-energy states, as we will do in several examples.

Further motivation for confining the search to spaces with such an (extended) module structure can be obtained from arguments in Ref. [15]. It was argued there that for a special Hamiltonian that has a gap in the bulk spectrum (and no other assumptions), the space of zero energy (or edge) states in the limit of large sizes forms a chiral conformal field theory, with an action of both the Virasoro and $U(1)$ current algebras. However, it is not clear if all the operators that generate these algebras (or of the subalgebras that we use in this paper) must have the one-body form that we encounter below, even though certain of them, which relate to translations and rotations of all the particles, do. (The arguments of Ref. [15] also lead us to expect that the action of the current and Virasoro generators must be local in position space; it may be that this further condition implies that the one-body form must hold.) Nonetheless, such a form would seem natural, and the assumption that there is this module structure, which we will make even for cases that we don’t expect to be gapped in the bulk, and in finite size, will ensure that when the bulk is gapped (the cases of most interest to us anyway), this physics will emerge.

In the spaces of amplitudes produced by a MR construction, the entanglement subspaces automatically form a module over the symmetric polynomials—corresponding to the presence of a $U(1)$ current algebra in the CFT—and over the part of the Virasoro algebra mentioned above, and perhaps also over some larger algebra, depending on the CFT considered. In this set-up, the “missing” polynomials (corresponding to the orthogonal complement of an entanglement subspace) are related to a finite-size version of the presence of singular (also called null) vectors in the underlying Verma modules of the CFT, a topic familiar in CFT. If we understood these orthogonal complement spaces, we would be able to find projection operators that could be included in a special Hamiltonian whose zero-energy spaces contain the given spaces of amplitudes. We would then be left with the problem of finding a *minimal* Hamiltonian such that the latter two sets of spaces actually coincide.

7. Superconformal and $M(3, p)$ examples

To illustrate the preceding ideas in a concrete manner, we introduce a family of examples. The original motivation for this work was to find examples of special Hamiltonians for which the zero-energy states are precisely the spaces of amplitudes obtained from a unitary rational

CFT; a guiding example was the tricritical Ising state [14] for which the ground state wavefunction is known [22]. Unfortunately we have not succeeded in finding a special Hamiltonian for this example. But Ref. [22] gives a family of ground state wavefunctions, one for each of a continuous family of special Hamiltonians, and those are labeled by one parameter c and involve three-body interactions only. These wavefunctions are obtained from the MR construction using the $\mathcal{N} = 1$ superconformal field theories, and include the tricritical Ising ground state at one special value of c , the central charge. (In general, these SCFTs are neither unitary nor rational.) The filling factor in these states is $\nu = 1/3$.

In Section II of this paper we analyze the zero-energy spaces for this family of special Hamiltonians (with no use of CFT), and find their dimensions. We also show that this family of models based on $\mathcal{N} = 1$ SCFT, which are expected to have gapless bulk excited states in the limit, are adjacent to a MR phase of bosons at $\nu = 1/3$, consistent with their being on a phase boundary. In Section III, we show that (except for $c = 0$, which has larger degeneracy) in the case of the plane geometry these dimensions agree in the infinite size limit with those of the generic $\mathcal{N} = 1$ superconformal field theory (SCFT) construction. The amplitudes constructed from the SCFT are guaranteed to be zero-energy states, and in fact we can eventually prove that (for $c \neq 0$) the zero-energy subspaces for any finite N are identical with the spaces of amplitudes of that SCFT for the same N (that is, not only the ground state wavefunction, but all wavefunctions in these spaces). Hence these spaces are also equal (at generic values of c , as we discuss in a moment) to the entanglement subspaces at N_A equal to this value N (as $N_B \rightarrow \infty$); the functions in, and the dimensions of, these spaces are then all known. These arguments are very general and can be applied to many other MR constructions, whenever the zero-energy states can be constructed and the dimensions of the spaces agree with those of the vacuum of the CFT in the large-size limit.

At the special point corresponding to the tricritical Ising model, these generic dimensions are strictly larger than those for the spaces of amplitudes in the latter. This is because at this value of c , the modules of the generic $\mathcal{N} = 1$ SCFT (which will be precisely defined later) contain singular vectors. The quotient space, obtained by setting the singular vectors to zero (or “modding them out”), yields a module that has no remaining singular vectors. The same phenomenon occurs at a countable infinity of other “rational” values of c , and is connected with the $\mathcal{N} = 1$ SCFT minimal models. In general, the SCFT minimal models (like the Virasoro minimal models) are rational but not unitary, however, there is a subset that are also unitary; among the latter, the tricritical Ising theory is the non-trivial theory of lowest c .

These facts have several consequences. First, at these rational values of c , the entanglement subspaces of the ground state, which as we will show always agree with the spaces of amplitudes in the MR construction using

the *quotient* or minimal model SCFT, have lower dimensions than those at generic c . This shows, as mentioned above, that the correspondence of dimensions of the entanglement and zero-energy spaces does not always hold. Hence, as c varies, some pseudoenergies in the entanglement spectrum must move off to infinity at these values of c , and this will occur even in finite size (sufficiently large, though how large will depend on the c value involved).

Second, at the rational values of c , the “missing” polynomials in the entanglement subspaces for some values of N_A allow us to add corresponding terms to the special Hamiltonian which then still annihilates the ground state. These terms depend on c and do not annihilate the ground state at generic c .

In order to determine, for a particular rational value of c , whether addition of such terms in the Hamiltonian produces zero-energy states that coincide with the spaces of amplitudes in the MR construction using the quotient SCFT at the same c , one must analyze the zero-energy states, and then if they are not the same as the spaces of amplitudes, add further terms to the Hamiltonian. This seems difficult to carry through in general. However, a remarkable paper by Feigin, Jimbo, and Miwa (FJM) [23] (whose algebraic approach is very similar to ours, though without the physical applications in view here), shows in effect that for one rational value of c , there is a three-body special Hamiltonian whose zero-energy subspaces are precisely the spaces of amplitudes, for any particle number N . In fact, their analysis applies to a sequence of MR constructions using Virasoro minimal models $M(3, p)$ [p is a positive integer, and not divisible by 3; $p = 4$ gives the MR state, and $M(3, 5)$ is the Gaffnian [24]]. These spaces of amplitudes have been discussed previously in relation to the FQHE, but the existence of a special Hamiltonian in these models was apparently overlooked (except for $p = 4, 5$, for which the Hamiltonians were known). We give the Hamiltonians for several of these in explicit form, and find all their zero-energy wavefunctions for all p for both the plane and sphere geometries; the multiplicities of these reproduce results in FJM.

8. Structure of the paper

The structure of this paper is as follows. In the remainder of this section, we review some of the earlier work that is relevant for our paper, and describe the issues in greater detail. The basic issues are dual: special Hamiltonians and subspaces of “nice” trial functions. This duality leads to a “chicken and egg” problem, in that a special Hamiltonian (the chicken) produces spaces of nice zero-energy states (the eggs), while in turn spaces of states that satisfy the compatibility conditions can be used to produce Hamiltonians that annihilate them. It is not obvious with which of these two types of objects one should begin. But as both types of analysis hinge on the entanglement subspaces, which correspond to (possible)

terms in a special Hamiltonian, we begin with those.

In Section II, we consider the wavefunctions determined by the (continuous family of) “vanishing conditions” on the relative variables for any three particles, which is related to the $\mathcal{N} = 1$ SCFT. The methods make no use of CFT, and give the complete, linearly-independent states for both the plane and sphere geometries. The count of states on the plane is interpreted later. We also argue that this model sits at a phase boundary adjacent to a MR phase.

In Section III, we begin to use methods from CFT. We describe features of the MR construction in the operator language of vertex operator algebras (VOAs). We prove a Theorem that implies that the functions obtained from the MR construction span a space isomorphic to the VOA. We show that the entanglement subspaces of the ground state in particle partition span all the amplitudes in the limit as $N_B \rightarrow \infty$ if the CFT is non-degenerate, but in degenerate cases span a smaller space. We show that in CFT-related examples in which the count of zero-energy states of a special Hamiltonian can be carried out, these numbers and the wavefunctions themselves agree with the amplitudes even in finite size N .

Finally, in Section IV, we show in an example how when the entanglement subspaces are smaller than the space of amplitudes of a CFT (or zero-energy states of a special Hamiltonian), because the CFT contains singular vectors, terms can sometimes be added to the special Hamiltonian such that agreement is restored with the quotient CFT. The examples in this section use the $M(3, p)$ minimal models.

In Appendix A, the proof of Theorem 1 is given.

B. Decomposition of trial wavefunctions

We consider the example of the Laughlin-Jastrow wavefunction [3]

$$\Psi_L(z_1, \dots, z_N) = \prod_{i < j} (z_i - z_j)^Q, \quad (1.1)$$

where we have dropped the ubiquitous factor that is a function of $|z_i|^2$ (which is Gaussian for the plane [3], rational for the sphere [11]) as they do not enter the argument here. The degree in each z_i is N_ϕ (the number of flux quanta covered by the particles),

$$N_\phi = QN - Q. \quad (1.2)$$

If we include quasihole factors $\prod_i (z_i - w_k)$ for each of a set of quasiholes at positions w_k ($k = 1, \dots, n$), then we know that each quasihole factor can be expanded in elementary symmetric polynomials

$$\prod_i (z_i - w_k) = \sum_{m=0}^N e_m(-w_k)^{N-m}, \quad (1.3)$$

where

$$e_m = \sum_{i_1 < i_2 < \dots < i_m} z_{i_1} z_{i_2} \dots z_{i_m} \quad (1.4)$$

($m = 1, \dots, N$) are the elementary symmetric polynomials in N variables, and for convenience we also define e_0 by $e_0 = 1$. The elementary symmetric polynomials generate the algebra of all symmetric polynomials in N , and do so *freely* (there are no linear relations between products of elementary symmetric polynomials). Then expanding the product of quasihole factors in powers of the w_k 's produces symmetric polynomials, and by using sufficiently many quasiholes we can eventually produce all the symmetric polynomials. [As the functions are symmetric under permutations of the w_k s, this expansion can be viewed as a double expansion in symmetric polynomials in the two sets of variables]. These functions which contain the Laughlin-Jastrow factor times arbitrary symmetric polynomials can be viewed as representing the edge excitations of the Laughlin disk.

The Laughlin wavefunction itself has a somewhat similar form as the quasihole factors. Thus if we divide the particles into two groups, say $\{1, \dots, N_A\}$ and $\{N_A + 1, \dots, N\}$ (and let $N_B = N - N_A$), then we may write the function in the form

$$\Psi_L(z_1, \dots, z_N) = \sum_r \psi_{Ar}(z_1, \dots, z_{N_A}) \times \psi_{Br}(z_{N_A+1}, \dots, z_N), \quad (1.5)$$

where r is an arbitrary index and the functions ψ_{Ar} and ψ_{Br} are symmetric in their arguments. If Ψ_L were replaced by a generic state (symmetric polynomial), the functions ψ_{Ar} obtained in this way would span the set of symmetric polynomials times ψ_L in N_A variables, at least in low degree (clearly the degree of ψ_{Ar} must be less than $NN_\phi/2$, the degree of the state considered). It is natural to wonder if this is the case for the Laughlin state. There are several variants of this question (in each case, initially with N_A fixed): (i) in one version, we consider all polynomials obtained as N_B increases to infinity; (ii) in a second, N_B is instead fixed, and it follows that the highest degree in any z_i with $i \in \mathcal{A}$ is bounded by N_ϕ [version (i) corresponds to the limit of this as $N_B \rightarrow \infty$]. In one further version, (iii) we require that N_B be fixed, and also that in the functions ψ_{Ar} the degree in each z_i in $\{1, 2, \dots, N_A\}$ be less than or equal to some bound less than N_ϕ , while the degree of ψ_{Br} in each z_i in $\{N_A + 1, \dots, N\}$ must be strictly greater than the same bound, similarly. In this case, an explicit symmetrization over permutations of the coordinates is required in order to yield a valid decomposition of Ψ_L in eq. (1.5) (it is not required in the others preceding). In each version, one may finally consider different N_A , and take the limits as $N_A \rightarrow \infty$. These decompositions can be applied in the same way to other trial wavefunctions also, and the limits can be studied as there is such a wavefunction for each N (or for each N divisible by some $k > 0$).

These decompositions of the Laughlin wavefunction are equivalent to definitions of entanglement that have been considered recently. For entanglement, we assume that the Hilbert space \mathcal{H} has a tensor product form,

$$\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \quad (1.6)$$

at least, within each sector of fixed N_A . If we represent vectors in each space by wavefunctions, then we can decompose a function ψ accordingly as

$$\psi = \sum_r \psi_{Ar} \psi_{Br}, \quad (1.7)$$

where ψ_{Am} (ψ_{Bm}) belongs to \mathcal{H}_A (\mathcal{H}_B). The question is to characterize the degree and nature of the *entanglement*, that is the extent to which ψ is not a tensor product. (This is rather too naive for systems of many identical particles, which because of symmetrization do not have a product form. However, once some particles have been assigned as part A and others as B , the overall symmetry of the state is broken, and the product form applies.) One version for QH states corresponds to further restricting the single-particle LLL Hilbert space according to the angular momentum of each particle, which corresponds to the degree in each z_i , so that some values appear only in part A , and the remainder in part B . This “orbital” entanglement [20, 21] corresponds to the third version above. The entanglement subspaces spanned by the ψ_{Ar} for some trial wavefunctions were found to be smaller than the spaces of all symmetric polynomials of the correct degree—this is essentially the phenomenon of the “entanglement gap” discovered in Ref. [21]. Other studies have considered “particle” entanglement [20], corresponding to the second version above. (A more formal extended definition of particle partition, in particular, was given in Ref. [25].) It has been conjectured that the dimensions of the entanglement subspaces in particle partition on the sphere agree with the counting of quasihole states for $N = N_A, N_\phi$, provided N_B is sufficiently large [26], however this does not seem to have been proved to our knowledge. Another version of the entanglement spectrum is obtained by considering a bipartition in position space—the real-space entanglement spectrum [25, 27–29], which we will not consider further here. The limit that has usually been considered for orbital and for real-space entanglement is that in which $N \rightarrow \infty$ with the ratio N_A/N fixed. The first version above ($N_B \rightarrow \infty$ with N_A fixed) has not been considered as much.

As mentioned above, in Section III of this paper, we obtain detailed statements about the entanglement spaces in particle (and also real-space) entanglement, in the first version above, the limit $N \rightarrow \infty$ with N_A fixed. We believe this is the simplest, or most basic, limit of all. That is, we can show that after taking $N_B \rightarrow \infty$, the functions ψ_{Ar} span a space isomorphic to the space of amplitudes in N_A variables, where the CFT used to construct the amplitudes is a quotient of the CFT used in

the construction, obtained by setting any singular vectors to zero. Moreover, as $N_A \rightarrow \infty$, the space of amplitudes in a MR construction becomes isomorphic to the space of states in the vacuum representation of the same CFT. We note that there has been recent progress in establishing such relations for real-space and particle partitions of trial wavefunctions [29], with which the present result has some overlap. However, the arguments presented there made use of generalized screening behavior to obtain results about the inner products of the wavefunctions, which yields much more information about the entanglement spectrum (not only dimensions of subspaces) than we obtain here. Such behavior is not expected to hold for trial wavefunctions obtained from non-unitary or non-rational CFTs. By contrast, the method used in the present paper is purely algebraic in that the quantum mechanical inner product on the wavefunctions is not used, and while it gives less information, it works for all cases, including non-unitary and non-rational ones.

C. Subspaces and Hamiltonians

As we know, the entanglement subspaces are the key to searching for a special Hamiltonian for a given set of wavefunctions. Several works have addressed this problem, beginning from Ref. [4]. Recently, a start has been made on systematically studying the possible special Hamiltonians for more than two-body interactions, and finding their zero-energy many-particle states [30].

It will be useful to describe the spaces of symmetric polynomials in N variables in more detail. Because the algebra of symmetric polynomials is freely generated by the elementary symmetric polynomials (or, in fact by some other alternative sets of N generators), the number of symmetric polynomials in each degree can be found easily. Generally, for a vector space V of symmetric polynomials that have multiplicities (dimensions) b_d in each degree d , we can define the character,

$$\text{ch}_q A = \sum_{d=0}^{\infty} b_d q^d \quad (1.8)$$

where q is an indeterminate (the series is a formal power series, which means that its convergence is not of interest). For the space Λ_N consisting of all symmetric polynomials in N variables, the character is

$$\text{ch}_q \Lambda_N = \prod_{n=1}^N \frac{1}{(1 - q^n)} = \frac{1}{(q)_N}, \quad (1.9)$$

where we have introduced the compact and standard notation,

$$(q)_m = \prod_{n=1}^m (1 - q^n). \quad (1.10)$$

Because the Hamiltonian should be translationally (and rotationally) invariant, it is useful to decompose the possible behavior for N particles into the center of mass and

some “relative” or “internal” wavefunction. The center of mass coordinate is $Z = \sum_i z_i/N = e_1/N$, and the center of mass wavefunctions are spanned by Z^m , $m = 0, 1, 2, \dots$. The center of mass angular momentum can be raised by multiplication by Z , and lowered by applying $\partial/\partial Z = \sum_i \partial/\partial z_i$. (In these expressions, the sums over particles are from $i = 1$ to N . We can treat the derivatives as acting on the polynomials part of the wavefunction only [31]). For the internal part, coordinates can be defined, and a space of functions of these coordinates orthogonal to the center of mass variable can be constructed. However, there is no particularly natural definition for these. In any case, the functions in the internal variables must be annihilated by $\partial/\partial Z$, which means that the polynomials are translation invariant—they are functions of differences $z_i - z_j$ only. The space of general wavefunctions is thus spanned by Z^m times polynomials of the differences of coordinates. (The space of translationally-invariant symmetric polynomials can be freely generated by a modified set of $N - 1$ elementary symmetric polynomials, given in Ref. [30].) For two particles, the latter have the simple form

$$(z_1 - z_2)^{m'} \quad (1.11)$$

where again $m' = 0, 1, \dots$, and now m' must be even for bosons (symmetric polynomials) and odd for fermions. There is only one polynomial for each even (resp., odd) degree. This is not typical; for $N > 2$, the number of linearly-independent polynomials in each degree increases gradually. The space of polynomials in Z has character $1/(1 - q)$, and so the character of the space of wavefunctions for the internal motion is

$$\prod_{n=2}^N \frac{1}{(1 - q^n)}. \quad (1.12)$$

For example, for $N = 3$ particles, we have

$$\frac{1}{(1 - q^2)} \frac{1}{(1 - q^3)} = 1 + q^2 + q^3 + q^4 + q^5 + 2q^6 + q^7 + 2q^8 + \dots, \quad (1.13)$$

and the coefficients have the pseudoperiodic property that if the degree increases by 6, the coefficient increases by 1.

Following earlier examples, we may attempt to describe some spaces of polynomials by specifying the behavior that is “allowed” for members of the space as some numbers N_A of the particles come together; these are sometimes termed “vanishing conditions”. As discussed above, we choose to require the spaces to be modules over the algebra of symmetric polynomials. It is not clear if this property is really necessary. Physically, multiplying by a symmetric polynomial generates an excitation of the charge sector only.

For $N_A = 2$, the translation-invariant modules over the symmetric polynomials can be completely determined. There is only one generator, $(z_1 - z_2)^2$, so any such module is spanned by the even powers $(z_1 - z_2)^{m'}$ as above,

for $m' = m, m + 2, \dots$, times a power Z^M , $M = 0, 1, \dots$. Thus the distinct translation-invariant modules are labeled by $m = 0, 2, \dots$, and the elements of each such module can be generated by $(z_1 - z_2)^m$ by multiplication by symmetric polynomials. These correspond to the distinct special or pseudopotential Hamiltonians for bosons that produce the Laughlin ground, quasihole and edge states as zero-energy states [4] (it is similar for fermions, if we consider modules consisting of antisymmetric polynomials). Cases that include conditions on more than two particles are much richer. They include the MR state for bosons at $\nu = 1$, which arises from requiring the polynomials to vanish if any three coordinates coincide. This corresponds to the degree 0 term in the above series, which represents the dimension of the relative-coordinate space of symmetric polynomials. Because there is no relative term at degree 1, the functions vanish quadratically as the three coordinates come together (in any fashion, e.g. first two, and then the third, or with any fixed ratio of pairwise separations). The more-recently proposed Gaffnian ground state, and its edge and quasihole states, arises if we require the functions to vanish as degree 3 when three coordinates approach the same point [24]. Both of these spaces of allowed behavior form a module over the symmetric polynomials.

In the examples considered in Ref. [30], the allowed behavior was defined by requiring that all relative motion with degree less than some bound was forbidden, and all greater was allowed. We note that this automatically produces a module over the symmetric polynomials. However, not all those examples led to truly “nice” behavior, with a unique ground state (lowest total degree polynomials) for N particles, and moderate growth of the number of allowed or zero-energy states at higher degree. Based empirically on the evidence of past cases, such behavior would be highly desirable when searching for new model QH phases of matter. Other than the recent Gaffnian [24], no such nice spaces of functions were identified in that work that had not been found previously. In further work [32], some four-body interactions were studied, and related to a MR construction with S_3 CFTs. The methods developed in the present paper can be applied to that case also.

At the higher degrees, for example degree 6 or larger than 7 for the three-particle case above, there are more possibilities. We may choose to allow, for three particles, any degree greater than 6, and only a one-dimensional subspace of the space in degree 6. The choice of a subspace means that there is now a *family* of spaces of functions (a one-parameter family in this example), or a family of Hamiltonians, each producing one of the spaces. We will study this example in detail in this paper. We emphasize that the possibility of such a parameter or parameters was not included in the original versions of the “thin-torus-limit”, “pattern-of-zeroes”, and “root state” approaches [33–39]. But the objects under study are vector spaces, which allow use of such linear combinations (when the dimension is greater than one), not discrete

objects such as sequences of integers. The latter cannot represent general *vectors* in a vector space, though they can represent *basis* vectors, such as the monomial basis for many-particle states in the lowest LL. We should add that later versions of some of these approaches have gradually included more terms in the Hamiltonian, corresponding to subleading terms in the thin-torus limit, and the explicit use of CFT constructions rather than direct “classification” of spaces of polynomials in the pattern of zeroes approach.

D. Moore-Read construction

In the MR construction [7], trial ground state wavefunctions are constructed as correlation functions in a two-dimensional CFT. More precisely, the functions are conformal blocks, or chiral parts of correlation functions, which depend only on z , not on \bar{z} , and take the general form

$$\langle 0 | \mathcal{O}_{-N} \prod_{i=1}^N a(z_i) | 0 \rangle \quad (1.14)$$

for ground state, where $|0\rangle$ is the vacuum of the CFT, and

$$a(z) = e^{i\varphi(z)/\sqrt{\nu}} \psi(z), \quad (1.15)$$

where $\varphi(z)$ is a free chiral scalar field, and $\psi(z)$ is a Virasoro primary field with Abelian statistics from some CFT [8, 9] (more precise requirements will be explained later). The coefficient of φ in each exponential is a $U(1)$ charge. \mathcal{O}_{-N} is an operator of charge $-N$ (so that overall neutrality is satisfied). In MR, \mathcal{O}_N was $e^{-i \int_{\mathcal{D}} d^2z \rho}$ for some uniform density ρ and domain \mathcal{D} , such that it carries total charge $-N$. By including additional fields inside the chiral correlator, it is possible to produce wavefunctions describing quasiholes as well; we will not require the explicit form of these in this paper.

The operator product expansion (ope), essentially the short-distance expansion of correlation functions, plays an important role in CFT. The properties of the operators a as their coordinates become equal determines the algebraic structure of the theory. This information is then related to the entanglement subspaces already discussed. We will provide very detailed connections between the two later in this paper.

There is a completely general duality, explained in Section III below, that allows *any* spaces of symmetric polynomials that are characterized by “vanishing conditions” (or equivalently as the zero-energy spaces for some special Hamiltonian) to be constructed in a similar form (again see Ref. [23], and also the earlier [40]). In an informal account, we introduce an infinite set of operators ξ_{-n} , $n = 0, 1, \dots$, all of which commute, and let $\xi(z) = \sum_{n>0} \xi_{-n} z^n$; they form an algebra \mathcal{A} . We suppose that there is a vacuum $|0\rangle$. Then the vector

space spanned by the vectors $\prod_{i=1}^N \xi_{-n_i} |0\rangle$ is dual to the space of symmetric polynomials in N variables (much as if each ξ_{-n} is a creation operator for a boson in the n th orbital in the LLL). But now we may also impose local, z -independent algebraic relations among the $\xi(z)$ ’s, such as $\xi(z)^2 = 0$, or $\xi(z)\partial\xi/\partial z = 0$. In general, each relation can be a linear combination of products of ξ and its derivatives all evaluated at z , and must be homogeneous both in ξ and in derivatives. Clearly, if such a relation is multiplied by any number of $\xi(z_i)$ at arbitrary positions, or if linear combinations are formed, other valid relations are obtained—thus the combinations that are to be set to zero form an ideal \mathcal{I} in the algebra \mathcal{A} with which we started. Then the symmetric polynomials that are dual to the quotient \mathcal{A}/\mathcal{I} acting on $|0\rangle$ must obey vanishing conditions as some number of z ’s come together, corresponding to these relations. Conversely, for any vanishing conditions (of the usual translation and rotation invariant form) determining spaces of symmetric polynomials, such an ideal \mathcal{I} can be found. So part at least of the structure of the CFT construction above is general [with $a(z)$ in place of $\xi(z)$]. It is less clear whether the additional structure that the polynomials form a module over the Virasoro algebra (which is automatic in the MR construction), can be shown to be required, though as mentioned above, the first at least seems to be necessary in order for many polynomials satisfying the vanishing conditions to exist. In any case, finding a set of relations that generate such an ideal \mathcal{I} corresponds to finding special Hamiltonians with a minimal set of terms. Then the search for special Hamiltonians takes on a truly algebraic form.

In recent work, it has been emphasized that the functions produced by the MR construction for a given CFT can be viewed using the operator formalism for CFT. Then the operators $a(z)$ are chiral vertex operators, acting in an auxiliary Hilbert space, which is that of the two-dimensional Euclidean (i.e. one space, one time dimension) field theory, not the “physical” Hilbert space of particles in two space dimensions. Then such a construction corresponds to a continuous version [29] of the popular idea of a matrix product state (MPS)—continuous, that is in the sense that z_i ’s are continuous variables, and a continuum, and in fact chiral, CFT is used. In particular, the (infinite-dimensional) Hilbert space of the CFT corresponds to the auxiliary space of the MPS construction. Closely related ideas are proving fruitful in on-going numerical work on the trial states [41, 42].

II. ANALYSIS OF A SPECIAL HAMILTONIAN WITH A CONTINUOUS PARAMETER

In Ref. [22], it was shown that there is a simple form for the (one-parameter family of) ground state wavefunctions in the MR construction applied to the $\mathcal{N} = 1$ SCFT [where the field $\psi(z) = G(z)$, the superconformal current], which satisfy a corresponding family of “vanishing

conditions" as three coordinates become equal. This result can be interpreted as giving the ground state wavefunction for corresponding special Hamiltonians, whose form can be explicitly determined. In this section we construct explicitly all the zero-energy wavefunctions for these Hamiltonians and enumerate them. We emphasize we can view this as a direct analysis of the spaces of functions determined by the given allowed behavior of the three-particle functions that is one of this family of forms (or of zero-energy states for a corresponding special Hamiltonian), with no use of CFT, other than that it motivated the parametrization in terms of the central charge c . Though the analysis is somewhat similar to other examples (such as those in Ref. [11, 43, 44]), the present example seems to be the simplest in which there is a continuously-varying parameter in the functions. The section is divided into two parts: the first gives results for the plane geometry, the second for the sphere with variable number of flux, which includes the plane results as a special limiting case.

A. Zero-energy states in the plane

We have seen that for three particles, the symmetric polynomials of degree six in the internal or relative coordinates span a two-dimensional space. Then we may consider as a Hamiltonian the sum of projection operators onto all states of any group of three particles with internal angular momentum (for the three) less than six, and onto one of the two linearly-independent states of internal angular momentum six. Using Ref. [22], we will parametrize the *allowed* behavior using first the four particle (non-symmetric) function

$$\chi(z_1, z_2; z_3, z_4) = Az_{13}^3 z_{24}^3 z_{14}^3 z_{23}^3 + z_{13}^4 z_{24}^4 z_{14}^2 z_{23}^2, \quad (2.1)$$

where $z_{ij} = z_i - z_j$ for all i, j , and $A = c/3 - 1$ where c and A can be complex numbers. As $z_1, z_2, z_3 \rightarrow Z$, this becomes

$$\chi \sim \chi_3(z_1, z_2; z_3)(Z - z_4)^6, \quad (2.2)$$

where

$$\chi_3(z_1, z_2; z_3) = \lim_{z_4 \rightarrow \infty} z_4^{-6} \chi(z_1, z_2; z_3, z_4) \quad (2.3)$$

$$= Az_{13}^3 z_{23}^3 + z_{13}^4 z_{23}^2. \quad (2.4)$$

When symmetrized over 1, 2, 3, this parametrizes the allowed (internal) behavior for three particles in relative degree 6 (they must vanish in relative degree less than six). Apart from overall normalization of the polynomial (which involves integration over all space, and depends on the geometry used), this covers most of the space of quantum states in the two-dimensional space, which are uniquely parametrized by a point on the sphere \mathbf{CP}^1 . Namely, for finite complex $c \neq \infty$ we cover all except one pole. By first dividing by c , we can cover all except the opposite pole ($c = 0$) of \mathbf{CP}^1 .

We will first study the behavior of symmetrized functions when two coordinates coincide. (The exposition will be somewhat terse, as the approach is similar to that in the earlier physics literature, in Ref. [11, 43, 44].) For the three particle unsymmetrized function χ_3 , one finds that for $z_1 = z_2$ it is given by $(A+1)z_{13}^6$. This vanishes if $c = 0$. In this case, the analysis of the space of allowed functions is straightforward: allowed functions in $N > 2$ variables must vanish when any two coordinates are equal, so must be of the form $D_N f$ where $D_N = \prod_{i < j} (z_i - z_j)^2$ is the discriminant (the square of the Laughlin-Jastrow factor) in N variables, and f is an arbitrary symmetric polynomial. In this case the allowed (symmetrized) internal three-body behavior in degree six is the same as D_3 . The functions of this form are the Laughlin ground and quasi-hole states for bosons at $\nu = 1/2$, as described in his original paper. For $N = 2$, the allowed functions are all of the symmetric polynomials in two variables. This difference from the functions for the pseudopotential model may seem surprising, but it is allowed by the compatibility conditions mentioned in Section I A 3.

From here on, we restrict to $c \neq 0$ (with $c = \infty$ handled as described above). Then by referring to Ref. [22], we expect the ground state (the lowest degree polynomial of zero energy) to have filling factor $\nu = 1/3$. This is determined by the degree N_ϕ of the ground state in each z_i , and a number called the shift \mathcal{S} , which is 6, that is

$$N_\phi = 3N - 6. \quad (2.5)$$

Then $\lim_{N \rightarrow \infty} N/N_\phi = \nu$. For $c \neq 0$, the functions do not generally vanish when two particle coincide. We consider the map C_1 which sets two coordinates (without loss of generality, the last two) equal to Z_1 . It maps a polynomial from the space Λ_N of symmetric polynomials in N variables into the space $\Lambda_{N-2} \otimes \Lambda_1$ of symmetric polynomials in z_1, \dots, z_{N-2} times a symmetric polynomial in Z_1 . This map has a kernel, that is, the space of functions it annihilates. The functions in the kernel $\ker C_1$ are again of the form $D_N f$, for f an arbitrary symmetric polynomial (we write $D_N \Lambda_N$ for the space of such functions, and similarly in other examples). More generally, consider the maps C_m which set the last $2m$ coordinates equal in pairs, to Z_1, \dots, Z_m . That is, $C_m : f(z_1, \dots, z_N) \rightarrow f(z_1, z_2, \dots, Z_1, Z_1, Z_2, Z_2, \dots, Z_m, Z_m)$ and maps $\Lambda_N \rightarrow \Lambda_{N-2m} \otimes \Lambda_m$. Each of these has a kernel in Λ_N , and $\ker C_{m-1} \subseteq \ker C_m$.

We wish to describe the space \tilde{I}_N (which is a module over the algebra Λ of symmetric polynomials) that is annihilated by our special Hamiltonian for arbitrary fixed c . That is, functions in \tilde{I}_N vanish in a prescribed way in degree six as three particles come together, or faster. We define $F_m = \ker C_m \cap \tilde{I}_N$, for $m = 0, 1, \dots, \lfloor N/2 \rfloor$ (for $m = 0$, $C_0 = \text{id}$, with kernel $\{0\}$). This immediately gives a filtration (sequence of inclusions of subspaces into one another)

$$F_0 = \{0\} \subseteq F_1 \subseteq F_2 \subseteq \dots \subseteq F_{\lfloor N/2 \rfloor} \subseteq \tilde{I}_N = F_{\lfloor N/2 \rfloor + 1}. \quad (2.6)$$

If we consider $c = 0$, then $F_m = \tilde{I}_N$ for $m > 0$. On the other hand, for $c \neq 0$, all the inclusions are strict, as can be seen because χ_3 is nonvanishing in this limit (or maybe better seen below).

The first kernel is $F_1 = \tilde{I}_N \cap D_N \Lambda_N$. But as we saw above, the polynomials in \tilde{I}_N have three-particle relative behavior in degree six that is not of the form D_3 unless $c = 0$. For $c \neq 0$ then, polynomials in the kernel F_1 have no term of degree six (or lower) in the differences among any three coordinates, and so must be of the form D_N times a symmetric polynomial that vanishes when any three coordinates are equal. But the symmetric polynomials that vanish when any three coordinates are equal defines a module \tilde{I}_N^{MR} , that is the zero-energy states for the MR state for bosons at $\nu = 1$ [11, 17]. That means we can explicitly write down all functions in F_1 .

This generalizes to give a description of F_{m+1} , modulo F_m , for all m , including $m = \lfloor N/2 \rfloor$ if we let $F_{\lfloor N/2 \rfloor + 1} = \tilde{I}_N$. First, we easily see from the allowed behavior in \tilde{I}_N that, for each m , the image of any function in \tilde{I}_N (in particular, those in F_{m+1}) under C_m is divisible by

$$\prod_{i \leq N-2m} \prod_{k \leq m} (z_i - Z_k)^6 \prod_{1 \leq k < l \leq m} (Z_k - Z_l)^{12}. \quad (2.7)$$

Hence, now defining the induced map on the quotient space $C_m|_{F_{m+1}}/\ker C_m : F_{m+1}/F_m \rightarrow \Lambda_{N-2m} \otimes \Lambda_m$, which is injective by definition, then we find by a similar argument as the one just given (which was the case $m = 0$) that its image is contained in the space of polynomials of the form

$$\prod_{i < j \leq N-2m} (z_i - z_j)^2 \cdot \prod_{i \leq N-2m} \prod_{k \leq m} (z_i - Z_k)^6 \cdot \prod_{1 \leq k < l \leq m} (Z_k - Z_l)^{12} \cdot I_{N-2m}^{\text{MR}} \otimes \Lambda_m \quad (2.8)$$

for $m = 0, 1, \dots, \lfloor N/2 \rfloor$. (Here dots separate products of distinct scope, and we reuse indices in distinct products.) The functions in these spaces correspond to the “residues” in earlier papers [11, 44]. From this, it is easy to count the number of such states, as the dimensions for the modules $\tilde{I}_{N-2m}^{\text{MR}}$ are known. This is an upper bound, as we have not yet shown that a state exists giving each linearly-independent residue (that is, that the maps are surjective). It is clear that if they do exist, they are linearly independent.

To complete the argument first, we will construct one function that maps to each of the above residues. First (following Ref. [43]), we introduce a labeling of the N particles. The N particles are partitioned into $m_\alpha \geq 0$ ($\alpha = 1, 2, 3$) clusters of sizes r_α , where $r_\alpha = 2$ ($\alpha = 1, 2$), $r_\alpha = 1$ ($\alpha = 3$), so $N = 2(m_1 + m_2) + m_3$. One particle is assigned to each box in a Ferrers-Young diagram of at most two columns that corresponds to this partition. Then the particle coordinates will be written as $z_{ij}^{(\alpha)}$, where i, j label rows and columns respectively (as for a matrix i increases down the rows, while j increases to the right along the rows) of the rectangular

block consisting of m_α rows of length r_α . That is, the first m_1 rows form the first block, the following m_2 the second, and the remaining m_3 rows of length 1 form the third block. Thus the rows of the diagram are ordered with (α, i) above (α', i') , written $(\alpha, i) > (\alpha', i')$, if either $\alpha < \alpha'$, or $\alpha = \alpha'$, $i < i'$. We will deal with the MR part at the same time as the other parts. It will turn out that $m_1 = m$ in the above residues, while m_2, m_3 play a similar role for the MR part. The assignment of particle coordinates to $z_{ij}^{(\alpha)}$ will be summed over in the end, restoring the symmetrization.

The wavefunctions for given m_α and sets of integers $n_l^{(\alpha)} \geq 0$ ($l = 1, \dots, m_\alpha$, and $\alpha = 1, 2, 3$) are

$$\begin{aligned} \mathcal{S}_z & \left\{ \prod_{(1,i) > (1,i')} \chi(z_{i1}^{(1)}, z_{i2}^{(1)}; z_{i'1}^{(1)}, z_{i'2}^{(1)}) \cdot \right. \\ & \times \prod_{\alpha'=2,3} \prod_{(1,i) > (\alpha',i')} \prod_{j=1}^{r_{\alpha'}} \chi_3(z_{i1}^{(1)}, z_{i2}^{(1)}; z_{i'j}^{(\alpha')}) \cdot \\ & \times \prod_{(1,m_1) > (\alpha,i) > (\alpha',i')} \prod_{j=1}^{r_\alpha} (z_{i,j}^{(\alpha)} - z_{i',j}^{(\alpha')})(z_{i,j+1}^{(\alpha)} - z_{i',j}^{(\alpha')}) \cdot \\ & \left. \times D_{N-2m_1}(\{z_{ij}^{(\alpha=2,3)}\}) \cdot \prod_{\alpha} \prod_{l=1}^{m_\alpha} e_l \left(\left\{ \sum_{j=1}^{r_\alpha} z_{ij}^{(\alpha)} \right\} \right)^{n_l^{(\alpha)}} \right\} \end{aligned} \quad (2.9)$$

Here $j+1$ is treated cyclically mod r_α , so $z_{i,r_\alpha+1}^{(\alpha)} = z_{i,1}^{(\alpha)}$. $D_{N-2m_1}(\{z_{ij}^{(\alpha=2,3)}\})$ is the discriminant in all the variables of types $\alpha = 2$ and 3 , the elementary symmetric polynomials $e_l(\{\sum_{j=1}^{r_\alpha} z_{ij}^{(\alpha)}\})$ are in the variables $\sum_{j=1}^{r_\alpha} z_{ij}^{(\alpha)}$, one for each row of type α . To see that these belong to \tilde{I}_N , first notice that when two coordinates are equal, the polynomial under the symmetrizer \mathcal{S}_z vanishes unless they both have $\alpha = 1$ and i is the same, that is they are members of the same pair, which we term an “unbroken pair” when $\alpha = 1$. When three coordinates approach the same value, the function either tends to zero as $\mathcal{S}\chi_3$ in these variables, which is the allowed form prescribed, or else vanishes faster. If coordinates are set equal in pairs until there are m pairs and doing so again would make the polynomial vanish, then all these pairs are unbroken ($\alpha = 1$ for each) and $m = m_1$. Further, the resulting function (consisting of “half-broken” pairs with $\alpha = 2$, and “unpaired” particles with $\alpha = 3$) has the form of (2.8), in which a polynomial in $I_{N-2m_1}^{\text{MR}}$ can be identified using similar results from e.g. Ref. [43]. This completes the argument, and as a whole it shows that these functions form a complete and linearly-independent set. The ground (lowest degree) state, which corresponds to $m_1 = N/2$, $m_2 = m_3 = 0$, was found in Ref. [22], and has degree $N_\phi = 3N - 6$, as mentioned above.

The number of states for each degree (angular momentum) can now be derived either from the explicit basis, or from the residues, either for the whole thing or using results for $\tilde{I}_{N-2m_1}^{\text{MR}}$. The degree of each function, omitting

the symmetric polynomial factors, is

$$\begin{aligned} & 6m_1(m_1 - 1) + 6m_1(2m_2 + m_3) + 2m_2(m_2 - 1) \\ & + (2m_2 + m_3)(2m_2 + m_3 - 1) + 2m_2m_3 + m_3(m_3 - 1) \\ & = \frac{3}{2}N(N - 2) + 2m_2 + m_3 + \frac{1}{2}m_3^2 \end{aligned} \quad (2.10)$$

where we eliminated m_1 in favor of $N = 2m_1 + 2m_2 + m_3$. Then the character or generating function for the dimensions of polynomials at each degree in \tilde{I}_N is

$$\begin{aligned} \text{ch}_q \tilde{I}_N = & \quad (2.11) \\ q^{\frac{3}{2}N(N-2)} & \sum_{\substack{m_2, m_3 \geq 0: 2m_2 + m_3 \leq N, \\ (-1)^{m_3} = (-1)^N}} \frac{q^{2m_2 + \frac{1}{2}m_3(m_3+2)}}{(q)^{\frac{N-2m_2-m_3}{2}}(q)_{m_2}(q)_{m_3}}, \end{aligned}$$

where $(q)_m$ was defined in Eq. (1.10). This gives the complete counting of zero-energy states for our special Hamiltonian, for any finite number of particles in the plane.

As a check on $\text{ch}_q \tilde{I}_N$, we can calculate the character of \tilde{I}_3 , which should be all symmetric polynomials, except for those of degrees 0, 2, 3, 4, 5, and one at 6 in the relative variables (times center of mass factors). The formula for $\text{ch} \tilde{I}_3$ does agree with this; it can also be written as $\text{ch} \tilde{I}_3 = q^6(1 + q + q^2 - q^4 - q^5)/(q)_3$. Also $\text{ch}_q \tilde{I}_2 = 1/(q)_2$, as it should be.

We can take the limit $N \rightarrow \infty$ fixing attention on the zero-energy states of fixed degree relative to the ground state simply by taking the limit of this formula with m_2, m_3 fixed. That is

$$\begin{aligned} & \lim_{N \rightarrow \infty} q^{-\frac{3}{2}N(N-2)} \text{ch}_q \tilde{I}_N \\ & = \sum_{m_2, m_3 \geq 0: (-1)^{m_3} = (-1)^N} \frac{q^{2m_2 + \frac{1}{2}m_3(m_3+2)}}{(q)_\infty (q)_{m_2} (q)_{m_3}} \end{aligned} \quad (2.12)$$

The sum is still restricted in the parity of m_3 , which depends on whether $N \rightarrow \infty$ through even or odd values. This result has factored into separate factors for each m_α . The factor $1/(q)_\infty = \lim_{m \rightarrow \infty} 1/(q)_m$ is the limit of the character for the symmetric polynomials, any one of which may be multiplied into any function in the space, and in FQHE often represent edge excitations in the charge sector. The remainder of this expression will be interpreted in terms of CFT later.

An extension of what we did here would be to consider the states with some quasiholes at the center of the disk. For finite size these are merely special cases of the full set we obtained above. But when taking $N \rightarrow \infty$ we can obtain different limits, and then we get the characters for other sectors of the theory. In this way we can determine all the sectors without considering either the sphere or torus.

B. Location at a phase boundary

In the early stages of this analysis, we saw for $c \neq 0$ that if we require the functions to vanish when any two

coordinates coincide, then there is no possible allowed behavior in relative degree six (or less) for three particles. It followed that the allowed functions are of the form D_N times a function in \tilde{I}_N^{MR} . These are the zero-energy functions associated with the MR state at $\nu = 1/3$, and are independent of c . Hence, a special Hamiltonian with a two-body projection operator (with positive coefficient) in addition to the six three-body ones (including the single c -dependent one in degree six) already included produces for its zero-energy spaces the functions in $D_N \tilde{I}_N^{\text{MR}}$. The fact that there is a *continuous family* of Hamiltonians that produce $\nu = 1/3$ MR states for bosons does not seem to have been emphasized previously to our knowledge.

We will presume that the latter Hamiltonians generally have a gap in the bulk energy spectrum, and so lie in the MR phase (for any $c \neq 0$). But when the coefficient of the two-body term becomes zero, the zero-energy spaces are much larger, and the energy spectrum of bulk excitations should become gapless in the infinite-size limit, because for generic c this system corresponds to amplitudes from a non-unitary CFT. We conclude that, for any value $c \neq 0$, there is a phase transition when the coefficient of the two-body term is reduced to zero. The transition appears to have the same behavior for all non-zero c , as the zero-energy states have the same multiplicities, however their wavefunctions do depend on c . These states can be interpreted as containing varying numbers of the “unbroken pairs” which are more tightly bound than the “half-broken pairs” present in the MR ground state. When the coefficient of the two-body term becomes negative, the system presumably enters a phase different from MR, the nature of which is not obvious. However, we may speculate that its ground state consists of the more tightly bound pairs produced by the attractive two-body interaction, something like a strong-pairing phase. The presence of the phase transition is similar to that in some other examples discovered earlier [11, 45, 46].

C. Zero-energy states on the sphere

It is simple to repeat the earlier analysis of the residues ($z_1, z_2 \rightarrow Z_1$, etc) for $c \neq 0$ for the case of the sphere, in which the functions of z should be the same degree for each particle $i = 1, \dots, N$. For the residues, the functions take a Haldane-Laughlin-like form in the positions of clusters $Z_i^{(\alpha)}$, $i = 1, \dots, m_\alpha$ for $\alpha = 1, 2, 3$ (the $Z_i^{(3)}$ are simply the coordinates of the unpaired particles) and quasiholes $w_l^{(\alpha)}$, $l = 1, \dots, n_\alpha$ for $\alpha = 1, 2, 3$. Here the $w_l^{(\alpha)}$ appear in factors $\prod_{i,l} (Z_i^{(\alpha)} - w_l^{(\alpha)})$ for each α . When counting the degree of the wavefunction in each particle variable z_i , one must realize that each $Z_i^{(\alpha)}$ for $\alpha = 1, 2$ can come from either of two particles, so contributes only a half to the total degree in z_i . Then one

finds that the wavefunctions are of the same degree

$$N_\phi = (6N + n)/2 - 6 \quad (2.13)$$

(where $n \geq 0$ is even) in each z_i regardless of type α provided

$$n_1 = n, \quad (2.14)$$

$$n_2 = n - 4, \quad (2.15)$$

$$n_3 = \frac{1}{2}n_2 - m_3. \quad (2.16)$$

The last line follows from $n_1 = 2(m_3 + n_3) + 4$, and agrees with a corresponding relation in the MR state in the formulation of Ref. [44].

To write down the complete set of states such that one produces each residue, let us first note that the factors χ can be used to couple a pair of type 1 to a pair of type 2, instead of two factors χ_3 as used above. There is no problem viewing these as functions on the sphere. The factors $\chi_3(z_1, z_2; z_3)$ are still required to couple type 1 to type 3 (unpaired particles). They appear problematic as the two terms are not the same degree in z_1 and z_2 , and are of a different degree in z_3 . The first problem can be fixed by incorporating two quasihole factors directly into χ_3 :

$$\begin{aligned} &\chi_3(z_1, z_2; z_3; w_1, w_2) = \\ &A(z_1 - z_3)^3(z_2 - z_3)^3(z_1 - w_1)(z_2 - w_2) \\ &+ (z_1 - z_3)^4(z_2 - z_3)^2(z_2 - w_1)(z_2 - w_2). \end{aligned} \quad (2.17)$$

As $z_1, z_2 \rightarrow Z$, this tends to the χ_3 above for the plane, times $(Z - w_1)(Z - w_2)$. The lack of symmetry between w_1 and w_2 is not important, because of the eventual symmetrization over z_i 's. The relation $n_1 = 2(m_3 + n_3) + 4$ shows that there will always be more than enough quasiholes of type 1 to allow this to be done for all χ_3 factors that appear. χ_3 can be multiplied by additional factors $\prod_{l=1,2}(z_1 - w_l)(z_2 - w_l)$ to obtain the same degree in z_1, z_2 , as in z_3 , fixing the second problem also.

The wavefunctions on the sphere are

$$\begin{aligned} \mathcal{S}_z \left\{ \prod_{\alpha=1,2} \prod_{(1,i) > (\alpha,i')} \chi(z_{i1}^{(1)}, z_{i2}^{(1)}; z_{i'1}^{(\alpha)}, z_{i'2}^{(\alpha)}) \cdot \right. \\ \times \prod_{(1,i) > (3,i')} \chi_3(z_{i1}^{(1)}, z_{i2}^{(1)}; z_{i'1}^{(3)}; w_{2i'-1}^{(1)}, w_{2i'}^{(1)}) \cdot \\ \times \prod_{(1,m_1) > (\alpha,i) > (\alpha',i')} \prod_{j=1}^{r_\alpha} (z_{ij}^{(\alpha)} - z_{i',j}^{(\alpha')})(z_{i,j+1}^{(\alpha)} - z_{i',j}^{(\alpha')}) \cdot \\ \left. \times D_{N-2m_1}(\{z_{ij}^{(\alpha=2,3)}\}) \cdot \prod_{\alpha} \prod_{i,j} \prod_l^{(\alpha,j)} (z_{ij}^{(\alpha)} - w_l^{(\alpha)}) \right\}. \end{aligned} \quad (2.18)$$

The notation is identical to that for the plane, except for the introduction of the χ_3 above, and the final product of quasihole factors, in which $\prod_l^{(\alpha,j)}$ appears, where the

range of l depends on α, j :

$$\begin{aligned} l &= 2m_3 + 1, & \dots, & 2m_3 + n_3 + 2 & (\alpha = 1, j = 1), \\ l &= 2m_3 + n_3 + 3, & \dots, & n_1 & (\alpha = 1, j = 2), \\ l &= 1, & \dots, & n_2/2 & (\alpha = 2, j = 1), \\ l &= n_2/2 + 1, & \dots, & n_2 & (\alpha = 2, j = 2), \\ l &= 1, & \dots, & n_3 & (\alpha = 3, j = 1). \end{aligned} \quad (2.19)$$

Each of these functions produces just one of the residues when particles coincide in pairs. As the residues are symmetric under permutations of $w_l^{(\alpha)}$ (for fixed α), it is permissible to symmetrize over the $w_l^{(\alpha)}$ of each α in the full wavefunctions.

Now we calculate the character, which here is the group-theoretical character for the representation of the rotation group of the sphere acting on the zero-energy states, or in other words the generating function that counts the number of states of each L_z . We use the q -binomial coefficient in the form

$$\binom{m_\alpha + n_\alpha}{n_\alpha}_q = \frac{(m_\alpha + n_\alpha)_q!}{(m_\alpha)_q!(n_\alpha)_q!}, \quad (2.20)$$

where the q -factorial $(n)_q! = [n]_q[n-1]_q \cdots [1]_q$, and the q -deformed integers are

$$\begin{aligned} [n]_q &= q^{-(n-1)/2} + q^{-(n-3)/2} + \dots + q^{(n-1)/2} \\ &= \frac{q^{-n/2} - q^{n/2}}{q^{-1/2} - q^{1/2}}. \end{aligned} \quad (2.21)$$

(This differs from what was used in Ref. [44], but is more convenient for rotationally-invariant situations.) For $q \rightarrow 1$, the q -binomial reduces to the usual binomial coefficient. Then by counting the symmetric polynomials in the $w_l^{(\alpha)}$ that occur in the residues, the generating function is

$$\text{tr } q^{-L_z} = \sum_{m_1, m_2, m_3: 2m_1 + 2m_2 + m_3 = N} \prod_{\alpha=1,2,3} \binom{m_\alpha + n_\alpha}{n_\alpha}_q. \quad (2.22)$$

The results for counting states in the infinite plane can be recovered by letting $n \rightarrow \infty$, as follows. In the plane, by stereographic projection from the sphere, $-L_z$ corresponds to angular momentum in the plane (total degree), but we must shift $-L_z$ by adding $NN_\phi/2 = N(6N+n)/4 - 3N$. Combined with the highest L_z (which occurs for N even, $m_1 = N/2, m_2 = m_3 = 0$), which is $L_z = Nn/4$, we recover the degree of the ground state in the plane, $\frac{3}{2}N(N-2)$ (independent of n). After extracting this factor, the q -binomials take on the definition used in Ref. [44], in which $[m]_q \rightarrow (m)_q = (1 - q^m)/(1 - q)$, and this is convenient for taking limits such as $n \rightarrow \infty$ at fixed N . In this way the generating function (character) in the plane, Eq. (2.12), can be recovered.

Returning to the sphere, we can view the q -binomial in m_1, n_1 as describing the ‘‘positional’’ degeneracy of what would be the quasiholes if the bulk energy spectrum were gapped, as in a topological phase: there are $n = n_1$

quasiholes of charge $1/6$. The other factors describe the degeneracy for fixed positions. This depends on m_2, m_3 , while $m_1 = (2m_2 + m + 3)/2$ is the number of unbroken pairs. The total degeneracy for fixed positions is

$$\sum_{m_2, m_3} \binom{m_2 + n_2}{n_2} \binom{m_3 + n_3}{n_3} \quad (2.23)$$

where the sum is over m_2 and m_3 such that $N - 2m_2 - m_3$ is even and ≥ 0 , and n_2, n_3 are determined by n and m_3 as above. In a sensible phase, this would approach a finite limit as N (or N_ϕ) $\rightarrow \infty$ with n fixed. In particular,

$$\binom{m_3 + n_3}{n_3} = \binom{n_2/2}{m_3} \quad (2.24)$$

corresponds to m_3 fermions that can occupy $n_2/2$ states. This factor is never large, and m_3 must be less than or equal to $n_2/2 = n/2 - 2$. But the binomial coefficient in m_2, n_2 corresponds to m_2 bosons in $n_2 + 1$ orbitals, and as $N \rightarrow \infty$ with $n_2 > 0$ there is no upper limit on m_2 . That is, the number of half-broken pairs can be arbitrarily large as $N \rightarrow \infty$. Hence the total degeneracy for fixed positions of the quasiholes becomes arbitrarily large, similar to the permanent and Haffnian cases [11, 45]. This behavior cannot occur in a gapped (topological) phase of matter.

III. AMPLITUDES AND MODULES

A. CFT—operator methods

In this section, we first set up definitions and basic general arguments that will be used in the following. The starting point is the general form of the MR construction of trial wavefunctions, eq. (1.14) for the ground state. Then we describe our first main result.

In this paper, the main examples are paired states for which the operator product expansion (ope) of ψ reads

$$\psi(z)\psi(0) \sim \frac{1}{z^{2h_\psi}} + \dots \quad (3.1)$$

as $z \rightarrow 0$, where h_ψ is the conformal weight of ψ , and \dots denotes lower-order terms (not only non-singular terms). In our main examples, either (i) ψ is the field at the corner of the minimal block in a BPZ minimal model $M(p, p')$, that is $\psi = \phi_{(1,p)}$ or $\phi_{(p',1)}$ in the p, p' minimal model (we use the conventions of Ref. [9]), or (ii) ψ is the superconformal current $G(z)$ in an $\mathcal{N} = 1$ superconformal field theory. The first of these includes, but is more general than, the $\phi_{(2,1)}$ field used by FJM. (For the Laughlin states, the analysis is similar but easier.) We will, however, extend the general results to the case of generalized parafermionic CFTs, or clustered FQHE states, of order $k = 2, 3, 4 \dots$; in these CFTs, when the ope of ψ 's is iterated, k ψ 's are needed in order to obtain the identity operator on the right hand side, as

for the $k = 2$ case above (the case $k = 1$, where $\psi = I$, the identity operator, gives the Laughlin states only). In all cases, the right hand side of the ope of ψ with itself is assumed to contain only operators of the same scaling dimension as the operator in leading term, plus integers, and similarly for the iterated opes; ψ should be a “simple current” (these conditions ensure that, for suitable values of h and h_ψ , the wavefunctions we obtain are indeed just symmetric polynomials). The assumed behavior of the iterated ope's of ψ leads to restrictions on the value of h_ψ (we will not need the details). For paired ($k = 2$) cases, single-valuedness and symmetry implies that $\nu^{-1} - 2h_\psi$ must be a non-negative even integer. In particular, one can consider $\nu = 1/(2h_\psi)$ so the wavefunctions are non-vanishing when two coordinates coincide. For general k , we require that the conformal weight $h = \nu^{-1}/2 + h_\psi$ of a , be half a positive integer so that a and its adjoint generate a chiral algebra, or vertex operator algebra (VOA) [9, 50], as we will discuss below. In some cases $2h$ is odd, even though the wavefunctions that are produced describe bosons—the chiral algebra generated by a and a^* violates the spin-statistics theorem (this occurs only when ψ is part of a non-unitary theory, such as the case of the Gaffnian for bosons). We can also obtain antisymmetric wavefunctions for fermions in a similar way, either by relaxing these conditions (leading to chiral superalgebras, but still obeying or not obeying the spin-statistics relation), or simply by multiplying the wavefunctions for bosons by the Vandermonde determinant $\prod_{i < j} (z_i - z_j)$.

We have the charge current operator $j(z) = i\partial\varphi$, with operator product

$$j(z)e^{in\varphi(0)/\sqrt{\nu}} \sim \frac{n/\sqrt{\nu}}{z} e^{in\varphi(0)/\sqrt{\nu}}. \quad (3.2)$$

Thus j measures the charge, up to the factor $\sqrt{\nu}$. Being a CFT, the theory has a stress tensor

$$T(z) = T_\varphi(z) + T_\psi(z), \quad (3.3)$$

which is a sum of the stress tensors T_φ and T_ψ of the theories to which φ and ψ belong. We have $T_\varphi(z) = -\frac{1}{2}(\partial\varphi)^2$. Finally, we have a vacuum $|0\rangle$ of the combined CFT, which is the tensor product of the vacua of the φ and ψ theories, $|0\rangle = |0_\varphi\rangle \otimes |0_\psi\rangle$. We emphasize that in this section, all states and operators are those of the two- (1+1-) dimensional CFT, not the 2+1 dimensional many-particle system.

The mode expansion of a can be defined by

$$a(z) = \sum_{n \in \mathbf{Z}} a_n z^{-n-h} \quad (3.4)$$

as $z \rightarrow 0$ (acting to the right). The modes can therefore be extracted as

$$a_n = \frac{1}{2\pi i} \oint dz z^{n-1+h} a(z). \quad (3.5)$$

The integral is over a circle of radius r . As h is an integer, and there is no singularity at the origin if no non-trivial

operator is inserted there, we must have

$$a_n|0\rangle = 0 \quad (3.6)$$

for $n > -h$, and we can identify the state $a(0)|0\rangle = |a\rangle = a_{-h}|0\rangle$. All products of z -dependent operators are radially-ordered, unless otherwise stated. To obtain the product of modes in any given order, one uses the corresponding ordering of the radii of the circles in the integral defining each mode.

The mode expansion takes the same form for any conformal (Virasoro primary) field, with h replaced by the conformal weight, and the mode indices n must obey $n + h \in \mathbf{Z}$. In particular, the modes j_n of the weight-1 U(1) current $j(z)$, and L_n of the weight-2 stress tensor $T(z)$, obey the usual relations, in particular the Virasoro algebra relations for the modes $L_n = L_{n\varphi} + L_{n\psi}$ of T in radial quantization [9]. Also, the modes obey relations with $a(z)$,

$$[j_n, a(z)] = \frac{1}{\sqrt{\nu}} z^n a(z), \quad (3.7)$$

$$[L_n, a(z)] = h(n+1)z^n a(z) + z^{n+1} \frac{\partial a(z)}{\partial z}. \quad (3.8)$$

Finally, we also have

$$j_n|0\rangle = 0 \quad (n \geq 0), \quad (3.9)$$

$$L_n|0\rangle = 0 \quad (n \geq -1), \quad (3.10)$$

and we refer to the sets of modes in the last relations as the positive half of the current and Virasoro algebras, respectively.

For our main examples, in which $k = 2$, it is clear that the conjugate field of a (which creates an “antiparticle” from the vacuum, if a creates a “particle”) is

$$a^*(z) = e^{-i\varphi(z)/\sqrt{\nu}} \psi(z). \quad (3.11)$$

This construction works because ψ is its own conjugate. a^* has the same conformal weight h as a . Clearly a^* can be obtained by the local operation of ordinary complex conjugation, with $\varphi(z)$, $\psi(z)$ treated as *invariant* (self-conjugate).

We now assert that this operation can be viewed as the radial adjoint (here we differ slightly from some standard treatments, such as Ref. [9]):

$$a^*(z) = a(z)^\dagger. \quad (3.12)$$

For any conformal field, the radial adjoint is defined using radial quantization, and the radial coordinate plays the role of imaginary time (because of the Euclidean signature on the two-dimensional spacetime; note also that the definition of the adjoint operation depends on a choice of origin). That is, we first map conformally to the cylinder, with the axis of the cylinder as imaginary time; the map is $z \mapsto -i \ln z$. We then apply the adjoint operation, which strictly speaking is only the usual operator adjoint at $|z| = 1$ (imaginary time equal to zero); for

non-zero time, it involves reversing the sign of imaginary time (in addition to complex conjugation), as in the imaginary-time formalism in many-body theory [48]. That is, $-i \ln z$ is invariant in this step. Finally, we apply the inverse conformal map back to the plane. The conformal map to the cylinder means multiplying by a factor z^h , and by the inverse for the inverse map. Thus in this point of view, $z^\dagger = z^{-1}$, except in the z^{-h} factors which are invariant. Putting all these steps together, we have

$$a(z)^\dagger = \sum_n a_n^\dagger z^{n-h}. \quad (3.13)$$

In modes, using the standard form of mode expansion for a^* or a^\dagger , this then implies

$$(a^\dagger)_n = (a_{-n})^\dagger. \quad (3.14)$$

(This last formula is standard [8].) The latter formulas apply to all conformal fields, for example $\psi(z)$, and also to $j(z)$, $T(z)$ which are both self-conjugate, so that $j_{-n} = j_n^\dagger$, and so on.

Usually adjoints are defined relative to an inner product, but in CFT we frequently do the reverse. We take the preceding formulas for the adjoint operators, together with the inner product of the vacuum $\langle 0|0\rangle = 1$. Inner products of states defined as any finite collection of modes of the fields acting on $|0\rangle$ can now be defined using the preceding formulas, and using the commutation relations to simplify expectation values of the modes. This determines the inner product on the space of states, which is Hermitian but not necessarily positive definite or non-degenerate.

We define

$$|0_N\rangle = e^{iN\varphi(0)/\sqrt{\nu}}|0\rangle, \quad (3.15)$$

or in an obvious notation $|0_N\rangle = |0_{\varphi N}\rangle \otimes |0_\psi\rangle$. Also we can define an out vacuum of charge $-N$ by

$$\langle 0_N| = \lim_{Z \rightarrow \infty} Z^{N^2/\nu} \langle 0| e^{-iN\varphi(Z)/\sqrt{\nu}}. \quad (3.16)$$

[We can do similarly for out states of other conformal fields, using twice their conformal weight in place of N^2/ν , thus $\langle a| = \lim_{z \rightarrow \infty} z^{2h} \langle 0| a^*(z)$.] Then we have

$$\langle 0_N| 0_{N'}\rangle = \delta_{N,N'}. \quad (3.17)$$

We can now define the chiral algebra, or vertex operator algebra (VOA) [47], of our theory as the algebra generated by the modes of fields generated by repeated ops of $a(z)$ and $a^*(z)$. (We do not mean to imply that a chiral algebra and a VOA are the same concepts when defined rigorously, however for our purposes greater precision will not be required.) We will argue that it includes the current operator $j(z)$ and both stress tensor operators, $T_\varphi(a)$ and $T_\psi(z)$.

For the φ theory, we define the space of states \mathcal{F}_N ($N \in \mathbf{Z}$), to be the Fock spaces generated from $|0_{\varphi N}\rangle$ by

the action of the modes j_n ($n < 0$). For the ψ theory, we define the space M_0 (M_1) to be the space of states generated by the action of an even (odd) number of ψ 's on the vacuum $|0_\psi\rangle$. In the Virasoro minimal model case, M_0 is the vacuum $(1, 1)$ irreducible Virasoro module, and M_1 is the irreducible Virasoro module generated by $|\psi\rangle$. In the SCFT case, we will discuss the structure of the spaces later. Then, as in Ref. [23], the space V of all states spanned by a and a^* acting on $|0\rangle$ is $V = \bigoplus_N V_N$, where

$$V_N = \begin{cases} \mathcal{F}_N \otimes M_0 & (N \text{ even}), \\ \mathcal{F}_N \otimes M_1 & (N \text{ odd}). \end{cases} \quad (3.18)$$

The inclusion $V \subseteq \bigoplus_N V_N$ is clear, but the equality requires proof. First, we notice that V is a module over (i.e. a representation of) the VOA. By the state-operator correspondence (or the theory of VOAs) [8, 9, 47], V “has the structure of” the VOA, in particular there is one (linearly-independent) state for each non-zero, linearly-independent operator in the VOA, and the Virasoro level and the charge of the states and operators agree.

It is not too difficult to see that V as defined here is indeed generated by $a(z)$ and $a(z)^*$. Essentially, from the ope's one can extract $j(z)$, and hence separate out the spaces \mathcal{F}_N . The generalization from order $k = 2$ to $k > 2$ is straightforward, involving k modules M_0, \dots, M_{k-1} in place of the two above. A more detailed proof for all cases is given in Appendix A.

The mode operators a_n commute, because the operator product $a(z_1)a(z_2)$ is non-singular. The operators a generate a commutative subalgebra U (the principal subalgebra) of V , and the subspace spanned by these operators acting on $|0_N\rangle$ will be denoted V^N :

$$V^N = U|0_N\rangle \quad (3.19)$$

Then we have a chain of embeddings (injective maps)

$$\dots \longrightarrow V^4 \longrightarrow V^2 \longrightarrow V^0 \longrightarrow V^{-2} \dots, \quad (3.20)$$

all of which are subspaces of V . The embeddings arise because V^{N-2} includes $a(z_1)a(z_2)|0_{N-2}\rangle$, and the leading part as $z_1, z_2 \rightarrow 0$ gives $|0_N\rangle$, and hence all of V^N appears inside V^{N-2} . We call $W = V^0$ the principal subspace (it corresponds to the principal subalgebra). There is also [23] an operator that we will call \mathcal{Q} that shifts the charge by 2 (it is part of the operator construction of $e^{2i\varphi(z)/\sqrt{\nu}}$), and acts as

$$\mathcal{Q}a_n\mathcal{Q}^{-1} = a_{n+h} \quad (3.21)$$

on the modes. The map $\mathcal{Q} : V^N \rightarrow V^{N+2}$ is an isomorphism. \mathcal{Q} will be used later. The appearances of the number 2 in these statements should be replaced by k in general.

It was shown in Ref. [40] and FJM, for the particular set of models they consider, that V is isomorphic to the direct (or inductive) limit $\lim_{\longrightarrow} V^{-N}$ of the above chain

of embeddings, defined by

$$\lim_{\longrightarrow} V^{-N} \equiv \lim_{N \rightarrow \infty} V^{-N} \equiv \bigcup_N V^{-N}. \quad (3.22)$$

Their proof uses the irreducibility of V_N (V_N is the subspace of V of charge N) under the action of the algebra generated by the current $j(z)$ and the stress tensor $T(z)$. This holds in our minimal model constructions, of which the models in FJM are special cases. A more general version, for all $k = 2$ constructions, can be proved using the algebra generated by ψ itself, as can the generalization to all $k > 0$. That is, we have

Theorem 1: In notation as above, for the order k case,

$$V \cong \lim_{\longrightarrow} V^{-N}. \quad (3.23)$$

Equivalently, V is generated as a VOA by $a(z)$ and

$$e^{-ik\varphi(z)/\sqrt{\nu}}. \quad (3.24)$$

(The latter is needed to reach the negatively charged vacua $|0_{-N}\rangle$.) We again defer the detailed proof, which is not difficult, to Appendix A. This is the first main result of this Section. (For the Laughlin or $k = 1$ case, the Theorem is trivial, as V is generated by $a(z)$ and $a(z)^* = e^{-i\varphi(z)/\sqrt{\nu}}$, by definition.)

The principal subspace W contains vectors constructed from the commuting operators $a(z)$. W is a module (representation) of the principal subalgebra (i.e. itself, under the state-operator correspondence). From the relations (3.7), (3.8), we have commutation relations for the modes of $a(z)$,

$$[j_n, a_m] = \frac{1}{\sqrt{\nu}} a_{m+n} \quad (3.25)$$

$$[L_n, a_m] = [(h-1)n - m] a_{m+n}, \quad (3.26)$$

and see also eqns. (3.9), (3.10). W is thus also a module over these positive halves of the current and Virasoro algebras. This positive half of the Virasoro algebra includes the generators $L_0, L_{\pm 1}$ of the Möbius group, but the central charge never shows up. Note that L_{-1} generates translations.

We can describe the structure of W by considering an algebra \mathcal{A} spanned by products of the commuting indeterminates ξ_{-n} , $n = 0, 1, \dots$ (which correspond to a_{-n-h} by dropping the shift by h). The operators a in general obey further relations, and these relations generate an ideal $\mathcal{I} \subset \mathcal{A}$. Then we have the isomorphism

$$W \cong \mathcal{A}/\mathcal{I}. \quad (3.27)$$

We emphasize that the ideal \mathcal{I} of relations in \mathcal{A} can be thought of as generated by *local* relations among the a 's, because of locality in the CFT. Examples of such relations are $a(z)^2 = 0$ in the Laughlin $\nu = 1/2$ example, and $a(z)^3 = 0$ in the MR $\nu = 1$ example, in which ψ is a Majorana fermion field. The relations must be independent of position z (translation invariance), respect the degree

$d = \sum n_i$ of the products of ξ_{-n} s (rotation invariance), and respect the charge N (charge conservation). Each such subspace $\mathcal{A}_{d,N}$ of degree d and charge N is finite dimensional. Any such expression written as modes of $a(z)$ acting on $|0\rangle$ can be viewed as built from a 's and derivatives of a 's acting at $z = 0$. The translation invariance of the VOA allows such a relation to be moved to any point z . Moreover, there is a natural action of the (positive) current and Virasoro modes on \mathcal{A} , determined by the same relations (3.25), (3.26). The subspaces \mathcal{I}_N (of vectors with charge N) must be mapped into themselves by these modes.

B. Dual spaces of polynomials

We now turn to the “functional model” of W [23, 40] which corresponds to wavefunctions for bosons in the QH situation. We recall that the dual \mathcal{B}^* of a vector space \mathcal{B} is the space of linear maps from \mathcal{B} to the complex numbers, and the action of a map $d \in \mathcal{B}^*$ on a vector $b \in \mathcal{B}$ can be written as the “dual pairing” $\langle d, b \rangle$ (no complex conjugation is involved). The dual vector space of \mathcal{A} can be represented by symmetric polynomials; we write Λ_N for the symmetric polynomials in N variables (which are here usually written z_1, \dots, z_N), and $\Lambda_{d,N}$ for the subspace of polynomials of degree d . We have the natural dual pairing [23]

$$\langle f(z_1, \dots, z_N), \xi_{-n_1} \cdots \xi_{-n_N} \rangle = \text{Res}_{z_1=\dots=z_N=0} f(z_1, \dots, z_N) z_1^{-n_1-1} \cdots z_N^{-n_N-1}, \quad (3.28)$$

where f is a symmetric polynomial in Λ_N , and the residue is the usual one of complex analysis, applied to all z_i . That is, we simply pick out the part of f (expanded in the “monomial basis” of symmetrized products of powers of z_i) in which the degrees in the N distinct variables are (in any order) n_1, \dots, n_N . The dual space W_N^* of W_N (the subspace of W spanned by N a 's acting on $|0\rangle$) is then the subspace I_N of Λ_N of polynomials that annihilate the module \mathcal{I}_N with respect to the above pairing,

$$I_N = \{f \in \Lambda_N \mid \langle f, \xi \rangle = 0 \ \forall \xi \in \mathcal{I}_N\}. \quad (3.29)$$

This means that I_N can be viewed as the space of symmetric polynomials that satisfy certain “vanishing conditions” on their behavior as several z_i 's come to the same value; these conditions are derived from the local relations in \mathcal{I} .

Equivalently, we may think of I_N as the space of all symmetric polynomials constructed as

$$f_v(z_1, \dots, z_N) = \langle v, a(z_1) \cdots a(z_N) | 0 \rangle \quad (3.30)$$

where v is an element of the dual space V^* of V , and $\langle \cdot, \cdot \rangle$ is the dual pairing with V . Clearly such functions inherit vanishing properties from the relations satisfied by the $a(z)$ s. This form is very close to what we wish to use for wavefunctions, but it will be important later

to realize that the use of the *dual* space V^* here is not always the same as using the space of *out* states. The out states can be viewed as elements of the dual space, which act on the space V by the natural (Dirac) pairing. Thus the out space can be mapped into the dual space; however this map may not be injective. We will now discuss this.

If the inner product (Hermitian sesquilinear form) on V is non-degenerate, then we can identify the dual space V^* with V using the inner product (more precisely, we have a conjugate-linear map from one to the other). We recall that an inner product is degenerate (or singular) if there are vectors that have vanishing inner product with all other vectors; such vectors are called *singular* (or sometimes, mis-called null) vectors. The singular vectors form a subspace, and the space modulo the subspace of singular vectors (formed by “setting the singular vectors to zero”) possesses a non-degenerate inner product.

Instead of using the dual space V^* to obtain the polynomials, we can use the inner product and obtain polynomials

$$f_w(z_1, \dots, z_N) = \langle w | a(z_1) \cdots a(z_N) | 0 \rangle, \quad (3.31)$$

where $w \in V$. We denote by J_N the space of such polynomials for N a 's. Clearly, $J_N \subseteq I_N$. If the inner product is non-degenerate, then $J_N = I_N$. But if the inner product is degenerate, then J_N may not equal I_N , as we have already noticed above. In finite dimensional vector spaces, the space and its dual have the same dimension. The space I_N in each total degree d (we write these as $I_{d,N}$, and similarly for $J_{d,N}$) has the same dimension as W_N at L_0 eigenvalue $d + Nh$ (which we write as $W_{d,N}$). But $J_{d,N}$ has the same dimension as the quotient space, $W_{d,N}$ modulo singular vectors lying inside it. J_N can be identified as the dual of that quotient space.

The spaces of polynomials I_N and J_N at which we have arrived have further general properties that reflect general aspects of our construction. The right action of the current modes j_n ($n \geq 0$) and the Virasoro modes L_n ($n \geq -1$) on $v \in V^*$ give [23]

$$f_{v \cdot j_n}(z_1, \dots, z_N) = \frac{1}{\sqrt{v}} s_n f_v(z_1, \dots, z_N) \quad (3.32)$$

$$f_{v \cdot L_n}(z_1, \dots, z_N) = h(n+1) s_n f_v(z_1, \dots, z_N) + \ell_n(f_\Psi)(z_1, \dots, z_N) \quad (3.33)$$

where for $n \geq 0$

$$s_n = \sum_{i=1}^N z_i^n \quad (3.34)$$

are the sums of powers ($s_0 = N$), and for $n \geq -1$

$$\ell_n = \sum_{i=1}^N z_i^{n+1} \frac{\partial}{\partial z_i}. \quad (3.35)$$

These follow from eqns. (3.7), (3.8), (3.9), and (3.10). The sums of powers s_n (with exponents $0 < n \leq N$) are

another set of generators for the symmetric polynomials Λ_N , and so the space I_N is mapped into itself by multiplication by symmetric polynomials — it is a module over the algebra of symmetric polynomials. Similarly, I_N is a module for the positive half (actually $n \geq -1$) of the Virasoro algebra, acting by the ℓ_n 's. That is, $s_n I_N \subseteq I_N$ and $\ell_n I_N \subseteq I_N$. Both sets of operators raise the degree of the polynomial (for $n > 0$), and usually one might expect them to be associated with the raising (negative) parts of the algebras; the apparent reversal of sign is due to the dual relationships in use here. Identical arguments apply if the operators are applied (from the right) to the out-state vector $\langle w |$ to obtain an action on the functions in J_N . If we think of the adjoints j_{-n} and L_{-n} acting on the vector $|w\rangle$, then these are indeed raising parts.

It is useful to realize that, in terms of the interpretation of symmetric polynomials as wavefunctions in the LLL, ℓ_{-1} is a derivative with respect to the center of mass variable, while j_1 acts as multiplication by (N times) the center of mass. These obey the Heisenberg relation

$$[\ell_{-1}, j_1] = -N, \quad (3.36)$$

corresponding to the projective representation of the translation group of the plane.

Turning again to the question of singular vectors, for a degenerate inner product in the context of a CFT or VOA, the subspace of V (or of any other module obeying the adjointness conditions) consisting of all the singular vectors must form a submodule of the VOA. Conversely, in an indecomposable module of the VOA (i.e. one that is not a direct sum of modules), a submodule must consist entirely of singular vectors. Finally, the quotient of V by the subspace of singular vectors yields a non-degenerate VOA on which the inner product is non-degenerate. We note that a non-degenerate inner product is a necessary condition both for a rational and for a unitary CFT (but not sufficient for either).

In some situations, examples of which we will study in this paper, a VOA is first constructed that depends on a parameter, such as the Virasoro central charge c . For generic values of c , the space V may be non-degenerate. But for special “rational” values of the parameter, singular vectors may appear [8, 9], and then they have to form a submodule of the VOA inside V . The central point is that this subspace may intersect the principal subspace W . In terms of the algebra \mathcal{A} , there is then a larger ideal \mathcal{I}' (over \mathcal{A}) that corresponds to the singular vectors. That is $\mathcal{I} \subset \mathcal{I}' \subseteq \mathcal{A}$. Then $\mathcal{I}'/\mathcal{I} \subset \mathcal{A}/\mathcal{I} \cong W$ is the singular subspace of W . Then, as we know, J_N is the dual of the quotient of these spaces,

$$J_N \cong (W_N/(\mathcal{I}'_N/\mathcal{I}_N))^*, \quad (3.37)$$

or $J_N \cong (\mathcal{A}_N/\mathcal{I}'_N)^*$. In other words, the smaller space J_N is dual to a space that is modded out by additional (independent) relations among the a 's, described by \mathcal{I}' , rather than \mathcal{I} . That is, the subspace $J_N \subset I_N$ is defined as polynomials that obey additional (independent)

“vanishing conditions” as several z_i 's approach the same point, compared with those defining I_N if $J_N \neq I_N$. In particular examples, as a continuous parameter such as the central charge c is varied, this phenomenon may occur at particular “rational” values of the parameter.

C. Application to entanglement subspaces of ground state

Now we turn to the applications of the preceding ideas to QH trial wavefunctions. In this section we translate the preceding results, and use some additional ideas to derive statements about the entanglement subspaces of ground state (or other) wavefunctions obtained from a MR construction.

The starting point is to write the trial ground state wavefunction as in eq. (1.14), following MR. In that work the operator \mathcal{O}_{-N} was a uniform charge distribution, and it produces the Gaussian factors $e^{-|z|^2/4}$ in the wavefunctions in the plane (in the limit that the background charge distribution extends to infinity—note that this violates neutrality). These factors are part of the lowest Landau level wavefunctions (in the symmetric gauge), and are important when calculating quantum-mechanical inner products. In this paper we usually do not consider these inner products, and so the Gaussian may be dropped. It is then convenient to use instead the point charge operator

$$\mathcal{O}_{-N} = e^{-iN\varphi(Z)/\sqrt{\nu}} \quad (3.38)$$

and place it at $Z \rightarrow \infty$. This means that (the polynomial part of) the trial ground state wavefunction is given by the “conformal block”

$$\Psi(z_1, \dots, z_N) = \langle 0_N | a(z_1) \cdots a(z_N) | 0 \rangle. \quad (3.39)$$

More generally, we may then consider trial functions that are obtained as the “amplitudes” we have been discussing, that is functions in I_N (or J_N) for the CFT under consideration:

$$\Psi_v(z_1, \dots, z_N) = \langle v, a(z_1) \cdots a(z_N) | 0 \rangle. \quad (3.40)$$

The spaces of amplitudes I_N , obtained as v ranges over V^* (or J_N similarly, using out states) form spaces of nice trial wavefunctions.

The spaces I_N are dual to W_N . Using the map \mathcal{Q} , we can map $W = V^0$ to $V^{-N'}$, and similarly for W_N . Then Theorem 1 above implies that as $N' \rightarrow \infty$, the sum of spaces $\bigoplus_N I_N$ can be identified as dual to V itself. That is, the ground state for some $N = N'$ is viewed as the vacuum (approximate vacuum, for finite N), and other states, that differ from it in charge and degree by amounts that stay constant as the limit is taken, are viewed as excited; then finally take $N' \rightarrow \infty$. In other words, if we suppose that the space of amplitudes can be identified as a space of zero-energy wavefunctions,

and the latter can in turn be viewed as a space of edge states, we obtain the fact familiar in examples that the edge excitations (including those that change the particle number) of the ground state of a disk span the (dual of the) vacuum representation of the chiral algebra (VOA) used in the construction, as $N \rightarrow \infty$. (Such a relation of edge CFT with that used in the MR construction was proposed in MR [7].) In particular, the characters of the spaces agree, provided we subtract the charge (degree) of the ground state wavefunction from the total charge (degree) before taking the limit—this corresponds to the use of the operator \mathcal{Q} . This relation is one of those sometimes referred to as a “bulk-edge correspondence”.

Now let us consider the spaces J_N of amplitudes defined using the inner product on the CFT (for the moment, it may be the same space as I_N), and in particular view the ground state in this way, as in eq. (3.39). We let $N = N_A + N_B$, and consider N_B of the $a(z)$ s (say, those with $i = N_A + 1, \dots, N_A + N_B$) as acting to the left. This can be done by taking the absolute values of their coordinates z_i larger than all those of the z_i s with $i = 1, \dots, N_A$. Also, we may expand the former $a(z)$ in modes. Thus we arrive at out states of the form

$$\langle 0_N | \prod_{k=1}^{N_B} a_{n_k}. \quad (3.41)$$

Under the map (“adjoint”) from V to out states, these correspond to

$$\prod_{k=1}^{N_B} a_{-n_k}^* |0_N\rangle. \quad (3.42)$$

Comparing again with the first main result above, and replacing a in the argument by a^* (and reversing the sign of the charges), we learn that as $N_B \rightarrow \infty$, these states eventually span V . This means that applying this expansion to the ground state, the ground state decomposes into combinations of amplitudes in J_{N_A} , and as $N_B \rightarrow \infty$, *all of J_{N_A} is obtained*.

This decomposition in fact involves the entanglement subspaces. Dividing the particles into those with coordinates inside some disk of radius equal to one, centered at the origin, and those outside, is the definition of a real-space cut. We may sharpen the preceding argument by imagining that we insert a complete set of states in V ,

$$I = \sum_{v \in V} |v\rangle\langle v| \quad (3.43)$$

(for an orthonormal basis set of v) at radius one. Then we arrive by the preceding argument at the same conclusion, now for the entanglement subspaces in the limit as the size of part B tends to infinity. The dimensions of the subspaces in real-space entanglement are the same as those in particle partition. This was proved in Refs. [25, 27] by a linear-algebra argument. In the present context, it is nicer to explain it in terms of functions [28]:

the real-space cut leads to functions in N_A (N_B) variables restricted to lie in the disk, part A (or its complement for B). But these analytic functions can be analytically continued to the whole plane, giving the functions in the particle partition. Hence we have shown that as $N_B \rightarrow \infty$, the particle (real-space) entanglement subspaces of the ground state in a MR construction coincide with (are isomorphic to) the spaces J_{N_A} . This is the promised result concerning version (i) in the Introduction. Though for $N_A \rightarrow \infty$ this result can be extracted from the recent work [29], the proof here makes no use of the quantum-mechanical inner products on the polynomials (which required use of a generalized screening hypothesis), and consequently applies to a larger class of constructions; moreover, the result here holds in finite sizes N_A . In the *orbital* partition at fixed N_A , the entanglement subspaces are generally smaller than those in the other two forms, though it is possible that the dimensions become the same as the limit $N_A \rightarrow \infty$ is taken.

In the preceding two results, the use of distinct spaces I_N and J_N in each one was necessary in general, as we will see momentarily. But when the inner product on V is non-degenerate, the two spaces are identical. We then obtain a correspondence of the entanglement subspaces (as $N_B \rightarrow \infty$) with the spaces of amplitudes (or zero-energy states, or edge states) for $N = N_A$ particles. This is a finite-size form of bulk-edge correspondence.

Now we consider what happens if the inner product on V is degenerate, and there are singular vectors in W . The space of amplitudes I_N , to be viewed as spaces of “nice” functions, for the given CFT in the MR construction is a definition. For the entanglement subspaces of the ground state, it is the inner product in the CFT that is important, and hence the spaces J_N . This can be seen in several ways. For example, the use of a complete set of states as above presupposes a non-degenerate inner product. Or we may note that the same ground state function is obtained if in the MR construction one takes the quotient of the VOA by all singular vectors; the singular vectors do not contribute to correlation functions in the vacuum [8, 9]. A particular example is the $\mathcal{N} = 1$ superconformal model whose wavefunctions were studied in Section II. The ground state wavefunction can be determined by the superconformal algebra alone [22], regardless of whether the singular vectors are modded out.

This leads to an important consequence for such cases: if there are singular vectors in W , the entanglement subspaces J_N will generally be *smaller than* the corresponding space of amplitudes I_N (or zero energy states, when a special Hamiltonian exists), even as $N \rightarrow \infty$. This is true in particular for the superconformal examples at rational central charge. Some of the pseudoenergies (defined as minus the logarithm of the Schmidt singular values) in the Schmidt decomposition of the ground state must go to infinity as c approaches such a value. In this case the correspondence with the spaces of amplitudes (or “edge states”) is lost. However, the correspondence can be recovered by using instead the quotient CFT when finding

the amplitudes or zero-energy states. This will however involve a change in the special Hamiltonian, as we will now explain.

D. Zero-energy states and special Hamiltonians

In this Section, we begin to merge the streams of thought in this paper, applying the preceding analysis of this section to the problem of solving or constructing special Hamiltonians, using the $\mathcal{N} = 1$ SCFT model of Section II as a specific example. In that section, we saw that the short-distance ope behavior of the SCFT implies that the entanglement subspace for three particles in the ground state begins with one polynomial in degree six, followed by higher degrees, in the relative or internal variables; thus the other degree six, and the lower ones are missing. This means that imposing such vanishing conditions to be obeyed by the wavefunctions, or equivalently using a three-body special Hamiltonian that enforces such conditions on zero-energy states of the N -particle system, produces spaces of polynomials for each N that must contain the spaces of amplitudes of the SCFT, as obtained by the MR construction that we have discussed in this section.

It is not immediately clear whether the spaces of zero-energy states are exhausted by the spaces of amplitudes, or whether they also contain other linearly-independent states. This issue arises in most examples of special Hamiltonians, and we cannot always answer it. But in examples in which we can obtain all the zero-energy states, we can compare the counting of the states with that of the spaces of amplitudes. For the latter, we lack detailed results for finite N in general. However, it is a standard problem in CFT to compute the character of the vacuum module, which as we have seen corresponds to the $N \rightarrow \infty$ limit of the character of the spaces of amplitudes, and we can then compare these limits. With that information at hand, we will be able to show the equality of the spaces even for finite N .

Returning to the system of Section II, we have emphasized that the full characters include the counting of states at different charges from the vacuum. But it will be sufficient here to examine the limits for a sector of fixed charge $N' - N$, as $N' \rightarrow \infty$, because the interesting part is the ψ CFT. The result in eq. (2.12), with m_3 even, can be viewed as the sector of zero charge excited, relative to the vacuum (the other charge-even sectors differ only by a power of q , left over when removing the power $q^{\frac{3}{2}N'(N'-2)}$ for the ground state with N' particles, and the charge-odd sectors also involve summing over m_3 odd instead of even). The expression has factorized into $1/(q)_\infty$, which as we mentioned corresponds to excitations of the charge sector, and two other factors, each of which is a sum. They can be expressed in a different form as follows. We use combinatorial identities of Euler [49]. By specializing these general results, we can obtain

for the m_2 sum

$$\sum_{b=0}^{\infty} \frac{q^{2b}}{(q)_b} = \frac{(1-q)}{(q)_\infty}. \quad (3.44)$$

For the sum over m_3 , we first leave it unrestricted, and obtain two identities,

$$\sum_{f=0}^{\infty} \frac{(\pm 1)^f q^{\frac{1}{2}f^2+f}}{(q)_f} = \frac{\prod_{r=1}^{\infty} (1 \pm q^{r-\frac{1}{2}})}{1 \pm q^{1/2}}. \quad (3.45)$$

We may produce identities for the sums with m_3 even (odd) by taking half of the sum (difference) of the expressions with the plus and the minus.

If the simple factors on the right hand side are omitted (that is, $1 - q$ in the first, and $1 \pm q^{1/2}$ in the others), these products have familiar interpretations as the partition functions for chiral bosons and Majorana-Weyl fermions, respectively. But with those factors, a different interpretation is more useful.

Combining the three factors in the full charge-zero character, it will be convenient again to consider the unrestricted m_3 sum, weighted with $(\pm 1)^{m_3}$ (a “ \mathbf{Z}_2 graded” character). If we write \mathcal{V} for the space of states in the limit with the charge sector removed completely (the odd and even parts have to be combined carefully with the charge sectors, as we have seen), then we have

$$\text{ch}_q^\pm \mathcal{V} \equiv \text{ch}_{q,+1} \mathcal{V} \pm \text{ch}_{q,-1} \mathcal{V} = \frac{(1-q) \prod_{r=1}^{\infty} (1 \pm q^{r-\frac{1}{2}})}{(q)_\infty (1 \pm q^{1/2})}. \quad (3.46)$$

This character can be identified as that of the Kac module for the vacuum module of the $\mathcal{N} = 1$ superconformal algebra (in the Neveu-Schwartz sector). This is obtained as follows. The Verma module [8, 9] consists of all formal descendants of the identity under the positive modes L_{-n} , $G_{-n+1/2}$ of the stress tensor and superconformal current, respectively, where $n > 0$ is an integer. Using standard arguments (the Poincare-Birkhoff-Witt basis), the Verma module \mathcal{V}_0 of the identity is found to have the (graded) character

$$\text{ch}_q^\pm \mathcal{V}_0 = \frac{\prod_{r=1}^{\infty} (1 \pm q^{r-\frac{1}{2}})}{(q)_\infty}, \quad (3.47)$$

just as if all the modes L_{-n} , $G_{-n+1/2}$ for $n > 0$ (anti-)commuted (although they don't). Kac studied singular (“null”) vectors in such modules for the Virasoro and $\mathcal{N} = 1$ superconformal algebras, and found in particular that the vacuum module, for which the conformal weight is 0, has a singular vector at level 1 in the Virasoro sector, and level 1/2 in the superconformal current sector, for any value of the central charge c . The quotient by these singular vectors, which generate a submodule in the Verma module, leaves a module with exactly the character $\text{ch}_q^\pm \mathcal{V}$, and it is irreducible for generic values of c . That is, L_{-1} and $G_{-1/2}$ annihilate the vacuum in the quotient module. (For the Virasoro part, the fact that

the vacuum is annihilated by L_{-1} means it is translation invariant in the radial quantization point of view.) This conclusion is precisely what we expected according to Theorem 1.

To avoid confusion, we should emphasize that the VOA generated by $a(z)$ and $a(z)^*$, which describes the chiral algebra of the “edge theory” of our states (or would, if the bulk were gapped) is not superconformal; it does not contain a superconformal current of weight $3/2$, as one may see from the character above in the neutral sector (like all other half-odd-integer weights, it is removed from the character $\text{ch}^\pm \mathcal{V}$ by the projection to the even sector). Of course, the original superconformal current G still occurs, but only as part of a charged field a . For the basic case $\nu = 1/3$, this operator has weight 3 and is bosonic. The chiral algebra does contain a copy of the Virasoro algebra.

We can now draw a conclusion about the space \tilde{I} of zero-energy states of the special Hamiltonian. We know that it must contain the space of amplitudes I_N of the VOA as we have discussed. But the count of states shows that in fact, for any non-zero value of c , these spaces coincide at infinite N , in a suitable sense, that involves a shift in both the charge and degree that goes to infinity in the limit. This implies that the spaces I_N and \tilde{I}_N are the same in a certain limit of large N and large degree d . But we can now prove

Theorem 2: In the example of this section, the spaces of polynomials I_N (the amplitudes) and \tilde{I}_N (the zero-energy states) are the same for any particle number N . Sketch of proof: \tilde{I} is dual to \mathcal{A} modulo the ideal generated by the three-particle ideal $\mathcal{I}_3 = \mathcal{I} \cap \mathcal{A}_3$, which tells us what combinations of ξ_{-n} ’s to set to zero. In other words, it is defined as all those symmetric polynomials (in any number of variables) satisfying the vanishing conditions on three particles. Suppose the ideal \mathcal{I} contained some further elements not in the ideal generated by \mathcal{I}_3 , and that these first occur for $N = N_1 \geq 3$. Equivalently, for $N \geq N_1$, I_N would require some further vanishing conditions not accounted for in \tilde{I}_N . For such N the dimension $\dim I_{N,d}$ would be smaller than $\dim \tilde{I}_{N,d}$ at some d . But for any N , the ideal \mathcal{I}_N is translationally invariant, $L_{-1}\mathcal{I}_N \subseteq \mathcal{I}_N$; its generators are local expressions in a and its derivatives, evaluated at z , and the corresponding relations hold for all z . In other words, the vanishing conditions involve only *differences* of N_A of the coordinates, not the center of mass of these (the differences are the relative or internal variables that appeared earlier). If for any N , $\dim I_N$ is smaller than $\dim \tilde{I}_N$ at some degree due to additional vanishing conditions, these affect only the internal variables, and so the effect propagates to all larger degrees because of the center of mass degree of freedom, which contributes $(1-q)^{-1}$ to the character. It also propagates to larger N , because in terms of \mathcal{I} , any additional relations can be simply multiplied by $a(z)$. Consequently the dimensions would obey $\dim I_{N,d} < \dim \tilde{I}_{N,d}$ (strict inequality) for all asymptot-

ically large N and d . This contradicts the fact that the asymptotics of the two are known to be the same. Hence the ideal \mathcal{I} is generated by \mathcal{I}_3 , and the spaces of amplitudes I_N coincide with the spaces of zero-energy states \tilde{I}_N for all N . The characters of the components of the principal subspace are $\text{ch}_q W_N = \text{ch}_q \tilde{I}_N$, which we have calculated. (For W_N , we define the character unconventionally, as $\text{ch}_q W_N = \text{tr}_{W_N} q^{L_0 - N h}$.)

We want to emphasize that the technique used to prove Theorem 2 is very general. It can be applied to other examples in which some translation-invariant relations involving some number of a ’s and their derivatives are known, or equivalently given a special Hamiltonian, if two conditions are met. These are that we can count the number of zero-energy states in each degree d and particle number N , and that these numbers agree with the vacuum module V of the CFT (or VOA) asymptotically in the standard limit for dealing with the edge states. When these conditions are met, the equality then extends to finite N . We point out that it has been a perennial problem when using the MR construction to show that the zero-energy states of some special Hamiltonian are precisely the spaces of amplitudes, and no larger. For the case of the RR states, the proof was constructed earlier [40], but to the present authors it appears technically difficult, and it may not be possible to extend it to other cases. By contrast, the argument given here deals with the MR and RR states using only the counts of states that have long been known. In FJM, the same result is reached in their examples, but the last part of the argument is not explicitly given. Our approach also differs in that we have methods to find all the zero-energy states explicitly, and not only to give an upper bound. These methods were completed for the RR states by the authors of Ref. [43], but they quoted the old result of Ref. [40] to obtain the equality of the spaces of polynomials, rather than obtaining that as a byproduct of the construction.

Next we turn to the inverse problem, finding special Hamiltonians such that the zero-energy subspaces are some desired spaces of functions, such as the amplitudes I_N or J_N of a given MR construction. This is essentially the problem of finding “vanishing conditions” that characterize these spaces. If the spaces of amplitudes I_N or J_N can be defined as symmetric polynomials that satisfy some “vanishing conditions” as z_i ’s come to the same point, we may construct a Hamiltonian that annihilates these functions and gives positive eigenvalues to functions not in the subspace, by forming projection operators that project onto functions (or states in the quantum mechanical Hilbert space, of lowest LL states of N particles) that are *not* in the subspace. Now the amplitudes form a subspace, and functions linearly independent from them lie in the quotient space by that subspace. There is no natural subspace of representatives for this quotient space, in the absence of an inner product. In the quantum mechanical situation, we may use the L^2 inner product on lowest LL states of the plane, sphere, or torus, depending on our own interest. Hence

the form of the projection operator, onto some functions (in $N' = 2, 3, \dots$ variables) linearly independent from the “allowed” subspace, necessarily depends on the system geometry, and even on its size (N_ϕ). Hamiltonians that are a sum of such terms (necessarily invariant under permutations) with positive coefficients (the magnitude of the coefficients is not important for our discussion) have been in use in the QH effect for some time (see e.g. [4, 10, 11, 13]). Because the relations to be imposed on the functions are local, the projection operators in these special Hamiltonians are short range, even if they involve interactions among more than two particles.

In the known examples, the interactions represented by the projection operators involve a bounded number of particles, and bounded degree for the relations (total number of derivatives appearing in the vanishing conditions) for each number of particles. If these conditions are not met, then the special Hamiltonian would not be local. (Higher degrees, which must necessarily occur if more particles are involved, imply a more extended interaction). Actually, this need not be strictly true. We only require the projection operators to have positive coefficients, while the ground and quasihole states would be of zero energy. The higher degree, and so longer range, terms could be given numerically smaller coefficients, and then the Hamiltonian can be relatively short range (or effectively involve power laws in distance). But it seems preferable in aiming for physical examples that the boundedness in particle number and degree of the terms be respected as far as possible.

Because of the duality discussed in Sec. IIIB, we can describe a set of spaces of polynomials (one space for each N) in terms of the algebra \mathcal{A} and a translationally-invariant ideal \mathcal{I} , with the action of the positive currents modes (and hence, symmetric polynomials) and the translation generator as before, even in the absence of a MR construction from a CFT. (It is less clear if we should also insist on an action of the positive half of the Virasoro algebra.) Then the space of polynomials that satisfy the vanishing conditions obeys the duality $\tilde{\mathcal{I}} \cong \mathcal{A}/\mathcal{I}$. This is the point of view we will take for a while.

In terms of the discussion of the spaces $\tilde{\mathcal{I}}$, and the relations \mathcal{I} in \mathcal{A} , this means that the former are determined by a set of translation-invariant vanishing conditions, that should preferably be bounded in both N_A and d , and consequently finite. In terms of the relations in the ideal \mathcal{I} , this property admits a clear mathematical definition: there is a *finite* set of elements i_l (indexed by l), each i_l lying in a subspace \mathcal{I}_{d_l, N_l} , such that \mathcal{I} is generated from $\{i_l\}$ by taking linear combinations with coefficients in \mathcal{A} and using the action of positive current modes, and the translation generator L_{-1} . (We should point out that L_{-1} acts on ξ_{-n} in \mathcal{A} in the same way as on a_{-n} , except for the shift in index by h , and so raises the total degree of excitation $\sum_i n_i$, as it is the transpose of the derivative operator in the dual space of symmetric polynomials.) It is far from clear if this finiteness property will generally be the case for the VOAs of interest

here. This question now becomes an important issue to be studied in future work.

To describe the structure of the special Hamiltonians in practice, some additional notation will be useful. To present the information about what terms appear in the special Hamiltonian, we can form a character for the part involving interactions among N_A particles, in which the coefficients are the dimensions of the space onto which the projection operators project in each degree. For example, for the special Hamiltonian for the generic SCFT model, the series is

$$1 + q^2 + q^3 + q^4 + q^5 + q^6 \quad (3.48)$$

for the three-body terms, and zero for $N_A \neq 3$. This series is the character of the space \mathcal{I}_3 , omitting the factor $(1 - q)^{-1}$ which describes the uninteresting and ubiquitous center of mass degree of freedom that relates to translation invariance. Comparing with the series in Eq. (1.13) for the internal part for the space of all symmetric polynomials, we see that the first term allowed is one of those in degree six, followed by all those of higher degree. As these series are indeed finite at least in the examples we will discuss, it is simpler to present this information than the character for the “allowed” functions in $\tilde{\mathcal{I}}$. (Recall that $\text{ch}_q \tilde{\mathcal{I}}_3 = \text{ch}_q \Lambda_3 - \text{ch}_q \mathcal{I}_3$.) Thus this type of series allows a succinct description of the terms in the Hamiltonian, though without giving the precise vector onto which projection is performed, which is important for the complete description when there is more than one, as occurs here for degree six for three particles. We seek to identify a set of generators (in the sense above) for \mathcal{I} , and the use of a corresponding term in the Hamiltonian for each one should produce as the zero-energy subspace $\tilde{\mathcal{I}}_N$ for all N .

Because we consider finite sets of generators, it makes sense to add generators one by one, that is to add additional translation-invariant terms to the special Hamiltonian. In addition we must preserve the property that \mathcal{I} is a module over the (dual action of the) symmetric polynomials or positive current modes, if we are to have any hope of finding polynomials that are zero-energy states for all N , or of agreeing with some MR construction from CFT. On \mathcal{I} , the current modes j_n act by lowering the excitation degree, as they are the transposes of the sums of powers s_n that raise the degree of the symmetric polynomials.

The special Hamiltonian for the SCFT model above, like other known simple examples, satisfies these properties in a basically trivial way. Further cases will require less trivial checks that the ideal is a module over the positive current modes. To find examples, we return to the observation that the space of amplitudes J_N , constructed from inner products, which correspond to the entanglement subspaces of the (very large) ground state, may be smaller than the space of amplitudes I_N constructed from the dual space, when the latter contains singular vectors. The “missing” vectors in J_N give us an opportunity to

add elements to \mathcal{I} , so that the missing vectors are those forbidden by the additional vanishing conditions.

For the $\mathcal{N} = 1$ SCFT, it is known [50] for

$$c = \frac{3}{2} \left(1 - \frac{2(p-p')^2}{pp'} \right) \quad (3.49)$$

where p, p' are integers, $p-p'$ is even, and p and $(p-p')/2$ are coprime, that there are additional singular vectors in the modules, in particular in the vacuum module. This implies that they will occur at some corresponding degree in the spaces I_N also, at least for sufficiently large N . Setting all singular vectors to zero results in the superconformal minimal models, denoted $SM(p, p')$. One case is the tricritical Ising model example, which however still seems to be difficult. Here we will pursue a simpler example, studied in FJM (though not from the SCFT point of view). Like the tricritical Ising model, the $M(3, 8)$ Virasoro minimal model (which has central charge $c = -21/4$) contains the $\mathcal{N} = 1$ superconformal current at a corner in the Kac table, $(2, 1)$ in the $M(3, 8)$ case, and hence $M(3, 8) = SM(2, 8)$. FJM showed that in this case, the ideal for three particles has the character

$$(q)_1 \text{ch}_q \mathcal{I}_3 = 1 + q^2 + q^3 + q^4 + q^5 + q^6 + q^8, \quad (3.50)$$

and that \mathcal{I}_3 generates \mathcal{I} . The corresponding vanishing conditions thus determine spaces $\tilde{I}_N = I_N$ of polynomials for each N which are those of the VOA of the SCFT model, with the singular vectors set to zero. Compared with the special Hamiltonian of the generic superconformal model, an additional translation-invariant term has been added in degree 8, for three particles, and nothing else. In the next section, we analyze this model, and some others, in detail. These models provide non-trivial examples of the structure we have just discussed.

IV. ANALYSIS OF SPECIAL HAMILTONIANS FOR $M(3, p)$ SERIES

In FJM [23], the example described at the end of the previous section was analyzed in detail. It is just one in a series in which the CFT used is the $M(3, p)$ minimal model, $p = 4, 5, 7, \dots$ (p is not divisible by 3), with the central charge of the $M(3, p)$ minimal model

$$c = 1 - \frac{2(p-3)^2}{p}. \quad (4.1)$$

The field ψ is the $(2, 1)$ field in the Kac table, with conformal weight $h_\psi = (p-2)/4$. We concentrate on the basic cases of bosons, for which the filling factor is again $\nu = 1/(2h_\psi)$. These examples of the MR construction have also been described as $(k, r) = (2, p-2)$ models; here as before $k = 2$ means they are paired states, while $r = p-2$ is the degree of the lowest non-vanishing allowed behavior for $k+1 = 2$ variables.

The analysis of FJM begins with the ground state wavefunction (symmetric polynomial) they call φ_3 . They

show that the space I_N is generated over the symmetric polynomials by this and $\ell_1(\varphi_3)$, and compute the character, as presented above for $M(3, 8)$. Taking the three-particle ideal \mathcal{I}_3 as determining vanishing conditions, they then analyze the zero-energy subspaces for all N . Because a bound on the characters of these agrees asymptotically with a so-called fermionic character formula derived from that for the $M(3, p)$ minimal model, they conclude that \mathcal{I}_3 generates \mathcal{I} , and $\tilde{I}_N = I_N$ for all N . They also calculate the characters for functions on the sphere, corresponding to counting quasihole states.

We will not repeat more details of FJM's analysis here. That is because the remaining methods do not obviously generalize. We do two things: we construct explicit sets of wavefunctions for all the zero-energy states, along similar lines as above for the generic superconformal model; and we describe the three-body interaction Hamiltonian for which these are the zero-energy states, for both the plane and the sphere. For completeness, we do the same for the other members of the $M(3, p)$ sequence.

A. $M(3, 8)$ wavefunctions in the plane

To find the wavefunctions, we make use of FJM's results. The vanishing conditions on the N -variable functions are incorporated by requiring that any function occurring in the particle-partition decomposition for any $N_A = 3$ variables must lie in the span of φ_3 and $\ell_1(\varphi_3)$ times symmetric polynomials, that is in $\tilde{I}_3 = I_3$. They use a filtration obtained as in Sec. II by setting variables equal in pairs. A key point is that in the present case, because of the additional vanishing condition, the kernel of this map applied to three particles lies not only in $D_N \tilde{I}_N^{\text{MR}}$, it lies in its subspace $D_N \tilde{I}_N^{\text{Gf}}$, where \tilde{I}_N^{Gf} is the space of symmetric polynomials that vanish as degree three or higher when any three variables coincide (the ground state or lowest degree polynomial in this space has been called the Gaffnian function [24]). These are related to the $M(3, 5)$ minimal model, and are again explicitly known [24].

Combining this observation with the construction of functions as before, we find a complete and linearly-independent set of polynomials in $\tilde{I}_N^{[8]}$, essentially by replacing elements of the ideal $\tilde{I}_{N-2m_1}^{\text{MR}}$, that was used before, with elements of $\tilde{I}_{N-2m_1}^{M(3,5)}$. Explicitly, with notation

as before, we replace (4.9) with

$$\begin{aligned} \mathcal{S}_z \Bigg\{ & \prod_{(1,i) > (1,i')} \chi(z_{i,1}^{(1)}, z_{i,2}^{(1)}; z_{i',1}^{(1)}, z_{i',2}^{(1)}) \\ & \times \prod_{\alpha'=2,3} \prod_{(1,i) > (\alpha',i')} \prod_{j=1}^{r_{\alpha'}} \chi_3(z_{i,j}^{(1)}, z_{i,j+1}^{(1)}; z_{i',j}^{(\alpha')}) \\ & \times \prod_{(2,i) > (\alpha',i')} \prod_{j=1}^{r_{\alpha'}} \chi_3^{\text{Gf}}(z_{i,j}^{(2)}, z_{i,j+1}^{(2)}; z_{i',j}^{(\alpha')}) \\ & \times \prod_{(3,i) > (3,i')} (z_{i,1}^{(3)} - z_{i',1}^{(3)})^2 \\ & \times D_{N-2m_1} \left(\left\{ z_{i,j}^{(\alpha=2,3)} \right\} \right) \cdot \prod_{\alpha} \prod_{l=1}^{m_{\alpha}} e_l^{n_l^{(\alpha)}} \left(\left\{ Z_i^{(\alpha)} \right\} \right) \Bigg\}. \quad (4.2) \end{aligned}$$

Here the (α, i) th cluster center of mass is $Z_i^{(\alpha)} = \frac{1}{r_{\alpha}} \sum_{j=1}^{r_{\alpha}} z_{i,j}^{(\alpha)}$, and the only difference from the generic SCFT case is in the third line, where we define

$$\chi_3^{\text{Gf}}(z_1, z_2; z_3) = (z_1 - z_3)^2 (z_2 - z_3), \quad (4.3)$$

$$\chi^{\text{Gf}}(z_1, z_2; z_3, z_4) = \chi_3^{\text{Gf}}(z_1, z_2; z_3) \chi_3^{\text{Gf}}(z_2, z_1; z_4). \quad (4.4)$$

Note that this only affects couplings between pairs of particles of type $\alpha = 2$, and between a pair of type two and a particle of type three.

The degree of each function, omitting the elementary symmetric polynomial factors, is now

$$\frac{3}{2}N(N-2) + m_2^2 + m_2 m_3 + \frac{1}{2}m_3^2 + m_2 + m_3. \quad (4.5)$$

Hence the character is identical to that given in FJM:

$$\begin{aligned} q^{-\frac{3}{2}N(N-2)} \text{ch}_q \tilde{I}_N^{[8]} = \\ \sum_{\substack{m_2, m_3 \geq 0: 2m_2 + m_3 \leq N, \\ (-1)^{m_3} = (-1)^N}} \frac{q^{m_2^2 + m_2 m_3 + \frac{1}{2}m_3^2 + m_2 + m_3}}{(q)^{\frac{N-2m_2-m_3}{2}} (q)_{m_2} (q)_{m_3}}. \quad (4.6) \end{aligned}$$

For $N \rightarrow \infty$ we can add the odd and even sectors to recover the $SM(2, 8)$ vacuum character in fermionic form [51]

$$\begin{aligned} (q)_{\infty} \text{ch}_q V_{(1,1)}^{SM(2,8)} &= (q)_{\infty} \left(\text{ch}_q V_{(1,1)}^{M(3,8)} + q^{3/2} \text{ch}_q V_{(2,1)}^{M(3,8)} \right) \\ &= \sum_{m_2, m_3 \geq 0} \frac{q^{m_2^2 + m_2 m_3 + \frac{1}{2}m_3^2 + m_2 + m_3}}{(q)_{m_2} (q)_{m_3}}, \quad (4.7) \end{aligned}$$

the first few terms of which are

$$(q)_{\infty} \text{ch}_q V_{(1,1)}^{M(3,8)} = 1 + q^2 + q^3 + 2q^4 + 2q^5 + 4q^6 + \dots, \quad (4.8)$$

$$(q)_{\infty} \text{ch}_q V_{(2,1)}^{M(3,8)} = q^{3/2} (1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + \dots). \quad (4.9)$$

These should be compared with those for the vacuum Kac module of the generic case.

B. $M(3, 8)$ wavefunctions on the sphere

As for the generic superconformal model, the wavefunctions on the sphere can be found with only slightly more effort. The degree in each coordinate variable to be $N_{\phi} = (6N + n)/2 - 6$, as this is determined only by the “leading order”/“ground state” clustering behavior parameters ($k = 2, r = 6$). Repeating the power counting in the residues of the wavefunctions (4.2), we find that the required number of quasiholes of each type are

$$n_1 = n, \quad (4.10)$$

$$n_2 = n - 2m_2 - m_3 - 2, \quad (4.11)$$

$$n_3 = \frac{n}{2} - m_2 - m_3 - 2. \quad (4.12)$$

The factors involving particles of type one are unchanged; we only need to homogenize the coupling (4.3) between a half-broken pair of type $\alpha = 2$ and a particle of type three:

$$\chi_3^{\text{Gf}}(z_1, z_2; z_3; w_1) = (z_1 - z_3)^2 (z_2 - z_3) (z_1 - w_1) (z_2 - w_1)^2. \quad (4.13)$$

Again, because $n_2 = 2n_3 + m_3 + 2$, there are always enough quasiholes of type two to do this. The basis functions on the sphere are therefore

$$\begin{aligned} \mathcal{S}_z \Bigg\{ & \prod_{(1,i) > (\alpha,i')} \chi(z_{i,1}^{(1)}, z_{i,2}^{(1)}; z_{i',1}^{(\alpha)}, z_{i',2}^{(\alpha)}) \\ & \times \prod_{(1,i) > (3,i')} \chi_3(z_{i,1}^{(1)}, z_{i,2}^{(1)}; z_{i',1}^{(3)}; w_{2i'-1}^{(1)}, w_{2i'}^{(1)}) \\ & \times \prod_{(2,i) > (2,i')} \chi^{\text{Gf}}(z_{i,1}^{(2)}, z_{i,2}^{(2)}; z_{i',1}^{(2)}, z_{i',2}^{(2)}) \\ & \times \prod_{(2,i) > (3,i')} \chi_3^{\text{Gf}}(z_{i,1}^{(2)}, z_{i,2}^{(2)}; z_{i',1}^{(3)}; w_{i'}^{(2)}) \\ & \times \prod_{(3,i) > (3,i')} (z_{i,1}^{(3)} - z_{i',1}^{(3)})^2 \cdot D_{N-2m_1} \left(\left\{ z_{i,j}^{(\alpha=2,3)} \right\} \right) \\ & \times \prod_{\alpha} \prod_{i,j} \prod_l^{(\alpha,j)} (z_{i,j}^{(\alpha)} - w_l^{(\alpha)}) \Bigg\}. \quad (4.14) \end{aligned}$$

The ranges of the quasihole factors in the last line are

$$\begin{aligned} l &= 2m_3 + 1, & \dots, 2m_3 + n_3 + 2, & (\alpha = 1, j = 1), \\ l &= 2m_3 + n_3 + 3, & \dots, n_1, & (\alpha = 1, j = 2), \\ l &= m_3 + 1, & \dots, (n_2 + m_3)/2, & (\alpha = 2, j = 1), \\ l &= (n_2 + m_3 + 2)/2, & \dots, n_2, & (\alpha = 2, j = 2), \\ l &= 1, & \dots, n_3, & (\alpha = 3, j = 1). \end{aligned} \quad (4.15)$$

It may be verified that the character on the sphere is still of the form

$$\text{tr } q^{-L_z} = \prod_{\alpha} \binom{m_{\alpha} + n_{\alpha}}{m_{\alpha}}_q, \quad (4.16)$$

with the n_α now given by (4.10)–(4.12); this is equivalent to theorem 5.14 in FJM (where the variables $N_\phi + 1$, N are denoted N , n). Taking the planar limit $n \rightarrow \infty$ with N fixed, we recover (4.6). Furthermore, the quasihole multiplicity for fixed positions is now

$$\sum_{\substack{m_2, m_3 \geq 0: \\ N - 2m_2 - m_3 \geq 0, \\ (-1)^{m_3} = (-1)^N}} \binom{n - m_2 - m_3 - 2}{m_2} \binom{n/2 - m_2 - 2}{m_3}, \quad (4.17)$$

the domain of which is clearly bounded and independent of N for $N > n$ (hence the sum itself also has these properties). Even though this multiplicity is now bounded, unlike in the generic superconformal case, we still expect this model to be gapless in the thermodynamic limit because it is a non-unitary CFT.

C. $M(3, p)$ series: general p

As found by FJM, the structure discussed above for the $M(3, 8)$ case holds for all $M(3, p)$ models with $p > 5$. That is, the kernel of each map C_1 for $\tilde{I}_N^{[p]}$ is $D_N \tilde{I}_N^{[p-3]}$ (and note that $\tilde{I}_N^{[4]} = \tilde{I}_N^{\text{MR}}$). Then the spaces of zero-energy states for the $M(3, p)$ models can be found recursively for all p not divisible by 3. Defining

$$s = \left\lfloor \frac{p}{3} \right\rfloor, \quad (4.18)$$

there will be $\alpha = 1, \dots, s$ types of particles with $r_\alpha = 2$, representing pairs in various stages of “brokenness,” and one type of unpaired particles with $\alpha = s + 1$, $r_\alpha = 1$. Schematically, the plane wavefunctions take the form

$$\begin{aligned} \mathcal{S}_z \left\{ \prod_{\alpha=1}^s \left[\prod_{(\alpha, i) > (\alpha', i')} \chi^{(\alpha)}(z_{i,1}^{(\alpha)}, z_{i,2}^{(\alpha)}, z_{i',1}^{(\alpha)}, z_{i',2}^{(\alpha)}) \right. \right. \\ \times \prod_{(\alpha, i) > (\alpha', i')} \prod_{j=1}^{r_{\alpha'}} \chi_3^{(\alpha)}(z_{i,j}^{(\alpha)}, z_{i,j+1}^{(\alpha)}, z_{i',j}^{(\alpha')}) \\ \left. \times D_{N-2 \sum_{\beta=1}^{\alpha} m_\beta} \left(\left\{ z_{i,j}^{(\beta=\alpha+1, \dots, s+1)} \right\} \right) \right] \\ \left. \times \prod_{\alpha} \prod_{l=1}^{m_\alpha} e_l^{n_l^{(\alpha)}} \left(\left\{ Z_i^{(\alpha)} \right\} \right) \right\}. \quad (4.19) \end{aligned}$$

All pairs of rows of type $1 \leq \alpha \leq s$ are coupled by a function

$$\begin{aligned} \chi^{(\alpha)}(z_1, z_2; z_3, z_4) = \\ \sum_{j=1}^{\left\lceil \frac{p-3\alpha}{2} \right\rceil} c_j^{(\alpha)} ((z_1 - z_3)(z_2 - z_4))^{p-3\alpha+1-j} ((z_1 - z_4)(z_2 - z_3))^j, \quad (4.20) \end{aligned}$$

where the $c_j^{(\alpha)}$ are constants which are partially fixed by the requirement that the symmetrization of this function reproduce the four-point conformal block:

$$\begin{aligned} \mathcal{S}_z \chi^{(\alpha)}(z_1, z_2; z_3, z_4) \\ = \langle a(z_1) \cdots a(z_4) \rangle_{M(3, p-3(\alpha-1)) \otimes U(1)}. \quad (4.21) \end{aligned}$$

Because these are Virasoro minimal models, the four-point block may be evaluated in terms of the ordinary hypergeometric function as [23]

$$\begin{aligned} \mathcal{S}_z \chi^{(\alpha)}(z_1, z_2; z_3, z_4) = ((z_1 - z_3)(z_2 - z_4))^{p-3\alpha+1} \\ \times {}_2F_1 \left(\alpha - \frac{p}{3}, 3\alpha - p - 1; 2\alpha - \frac{2p}{3}; x \right), \quad (4.22) \end{aligned}$$

where the cross-ratio $x = (z_1 - z_4)(z_2 - z_3)/(z_1 - z_3)(z_2 - z_4)$. The leading-order behavior in z_1, z_2, z_3 as $z_4 \rightarrow \infty$ then defines $\chi_3^{(\alpha)}(z_1, z_2; z_3)$.

We use the term “schematic” in referring to the wavefunctions (4.19) because $\chi^{(\alpha)}$ is not uniquely determined by Eq. (4.21): there exist non-symmetric functions of the form (4.20) which vanish under symmetrization but whose contributions do *not* vanish when used to construct a wavefunction of the form (4.19) with $N > 4$ particles. The ambiguity can of course be resolved by an examination of the singularity structure of the four-point block $\langle \psi(z_1) \cdots \psi(z_4) \rangle_{M(3, p-3(\alpha-1))}$, as was done for G in Ref. [22], but this must be done on a case-by-case basis as the conformal weight of ψ (and hence the number of singular terms in the $\psi \cdot \psi$ OPE) increases with p ; therefore this is the most we can say about the form of the wavefunctions without restricting ourselves to specific cases.

For general p , characters on the plane and sphere may still be computed without explicit basis functions; this was done in section 5 of FJM. We restate their results in a form more conventional in the FQHE literature [44, 52]: let

$$\left\{ \begin{matrix} n \\ F \end{matrix} \right\}_p \quad (4.23)$$

denote the quasihole degeneracy at fixed positions in the $M(3, p) \otimes U(1)$ theory on the sphere with n quasiholes and F not-fully-paired particles (in the notation of (4.19), $F = \sum_{\alpha=2}^{s+1} r_\alpha m_\alpha$). The relationship between N, n and N_ϕ is

$$N_\phi^{[p]} = \frac{p-2}{2} N + \frac{n}{2} - (p-2). \quad (4.24)$$

FJM’s equations (5.5), (5.6) then give a recursive expression for the quasihole degeneracy, analogous to the result obtained in Ref. [44] for \mathbf{Z}_k parafermions:

$$\left\{ \begin{matrix} n \\ F \end{matrix} \right\}_p = \sum_{F' \equiv F \pmod{2}} \binom{\frac{F-F'}{2} + n'}{n'} \left\{ \begin{matrix} n' \\ F' \end{matrix} \right\}_{p-3} \quad (4.25)$$

with $n' = n - F - 2$ and initial condition

$$\left\{ \begin{matrix} n \\ F \end{matrix} \right\}_p = \left(\frac{n}{2} + (p-4) \left(\frac{F}{2} - 1 \right) \right), \text{ for } p = 4, 5. \quad (4.26)$$

FJM solve this recursion in their theorem 5.14; in particular the numbers of quasiholes at each stage are

$$n_\alpha^{[p]} = n - 2(\alpha - 1) - \sum_{\beta=1}^{s+1} (\min(\alpha, \beta) - 1) r_\beta m_\beta \quad (4.27)$$

for $1 \leq \alpha \leq s$ and

$$n_{s+1}^{[p]} = \frac{n}{2} - s - \frac{1}{2} \sum_{\beta=1}^{s+1} (\beta - 1) r_\beta m_\beta + \frac{1}{2} ((p \bmod 3) - 2) (m_{s+1} - 2). \quad (4.28)$$

D. Explicit special Hamiltonians on the sphere and the plane

So far we specified the allowed behavior for any three particles as that belonging to the spaces $I_3^{[p]}$ for three particles. As these functions are explicitly known, we have only to construct translation-invariant projection operators onto the orthogonal complement of these spaces, and the sum of these any positive coefficients gives a special Hamiltonian that produces the allowed functions as zero-energy states. Unlike the allowed functions, the forbidden behavior, which must be orthogonal to the allowed, depends on the geometry of the system through the inner product in use. We give results for the sphere, for which the measure in stereographic coordinates (by mapping to the plane) is

$$\mu(z) d^2 z = \frac{dz d\bar{z}}{(1 + |z|^2/4)^{2(1+N_\phi/2)}}. \quad (4.29)$$

The results for the plane may be obtained by keeping only the leading order terms as $N_\phi \rightarrow \infty$.

FJM's results determine the character of the ideal $\mathcal{I}_3^{[p]}$ for the $M(3, p)$ case as

$$(q)_1 \text{ch}_q \mathcal{I}_3^{[p]} = \frac{(1 - q^{p-2})(1 - q^{p-1})}{(1 - q^2)(1 - q^3)}. \quad (4.30)$$

The series on the right-hand side terminate with a finite number of terms. The coefficients of these terms tell us how many orthogonal states must be forbidden in the polynomial behavior for each p .

As a basis, we use the translationally-invariant symmetric polynomials $\{\tilde{e}_n\}$, $2 \geq n \geq N$, defined in Ref. [30] in terms of the elementary symmetric polynomials e_n as

$$\tilde{e}_n(\{z_i\}) = e_n \left(\left\{ z_i - \frac{1}{N} \sum_{j=1}^N z_j \right\} \right). \quad (4.31)$$

We specify the orthogonal allowed and the forbidden behavior for each degree d in which the projection operator has non-zero image and kernel among the functions of that degree, since otherwise an arbitrary basis for that degree can be used (or omitted entirely). Recall that the lowest allowed degree is $r = p - 2$. The functions have not been normalized. The first examples are as follows:

$p = 7$ ($r = 5$): At $d = 6$, the allowed function is $9\tilde{e}_3^2 - 2\tilde{e}_2^3$; the forbidden (to be projected onto in the special Hamiltonian) is

$$27(9N_\phi^3 - 48N_\phi^2 + 73N_\phi - 40)\tilde{e}_3^2 + (63N_\phi^3 - 471N_\phi^2 + 1156N_\phi - 880)\tilde{e}_2^3. \quad (4.32)$$

$p = 8$ ($r = 6$): At $d = 6$, allow $\varphi_3^{[8]} \propto 9\tilde{e}_3^2 - \tilde{e}_2^3$; forbid

$$27(9N_\phi^3 - 60N_\phi^2 + 127N_\phi - 100)\tilde{e}_3^2 + 2(45N_\phi^3 - 408N_\phi^2 + 1199N_\phi - 1100)\tilde{e}_2^3. \quad (4.33)$$

The behavior at $d = 7$ is allowed, and at $d = 8$ allow $\tilde{e}_2\varphi_3^{[8]} \propto 9\tilde{e}_3\tilde{e}_2 - \tilde{e}_2^4$; forbid

$$54(9N_\phi^3 - 69N_\phi^2 + 160N_\phi - 140)\tilde{e}_3^2\tilde{e}_2 + (99N_\phi^3 - 948N_\phi^2 + 2999N_\phi - 3010)\tilde{e}_2^4. \quad (4.34)$$

$p = 10$ ($r = 8$): At $d = 8$, allow $\varphi_3^{[10]} \propto -18\tilde{e}_3^2\tilde{e}_2 + \tilde{e}_2^4$; forbid

$$27(9N_\phi^3 - 93N_\phi^2 + 316N_\phi - 392)\tilde{e}_3^2\tilde{e}_2 + (63N_\phi^3 - 786N_\phi^2 + 3217N_\phi - 4214)\tilde{e}_2^4. \quad (4.35)$$

At $d = 9$, allow $(\ell_1 - 16/3s_1)\varphi_3^{[10]} \propto 3\tilde{e}_3^3 - 2\tilde{e}_3\tilde{e}_2^2$; forbid

$$27(9N_\phi^3 - 84N_\phi^2 + 235N_\phi - 216)\tilde{e}_3^3 + (63N_\phi^3 - 804N_\phi^2 + 3397N_\phi - 4392)\tilde{e}_3\tilde{e}_2^2. \quad (4.36)$$

At $d = 10$, allow $\tilde{e}_2\varphi_3^{[10]} \propto -18\tilde{e}_3^2\tilde{e}_2^2 + \tilde{e}_2^5$; forbid

$$135(9N_\phi^3 - 102N_\phi^2 + 373N_\phi - 504)\tilde{e}_3^2\tilde{e}_2^2 + 2(117N_\phi^3 - 1542N_\phi^2 + 6745N_\phi - 9576)\tilde{e}_2^5. \quad (4.37)$$

Both behaviors at $d = 11$ are allowed, and at $d = 12$ we have a three-dimensional space; we must allow the two-dimensional space spanned by the (nonorthogonal) behaviors $\tilde{e}_3(\ell_1 - 16/3s_1)\varphi_3^{[10]} \propto 3\tilde{e}_3^4 - 2\tilde{e}_3^2\tilde{e}_2^2$ and $\tilde{e}_2^2\varphi_3^{[10]} \propto -18\tilde{e}_3^2\tilde{e}_2^2 + \tilde{e}_2^6$, and forbid the orthogonal complement

$$3645(81N_\phi^6 - 1755N_\phi^5 + 14607N_\phi^4 - 60117N_\phi^3 + 134284N_\phi^2 - 169756N_\phi + 110880)\tilde{e}_3^4 + 270(1053N_\phi^6 - 30834N_\phi^5 + 369873N_\phi^4 - 2330376N_\phi^3 + 8190988N_\phi^2 - 15458176N_\phi + 12418560)\tilde{e}_3^2\tilde{e}_2^2 + 2(20007N_\phi^6 - 641979N_\phi^5 + 8479917N_\phi^4 - 58925481N_\phi^3 + 226850956N_\phi^2 - 458053660N_\phi + 378655200)\tilde{e}_2^6. \quad (4.38)$$

None of the coefficients above factors as a polynomial in N_ϕ , and using different bases for the polynomials [such as Jack polynomials at the appropriate parameter value $\alpha = -(k+1)/(r-1)$] do not yield noticeable patterns in the expressions involved.

V. CONCLUSION

To conclude, we have explored the relations among several spaces of wavefunctions that can be constructed for a system of trial wavefunctions, especially when those can be constructed from a CFT. These were: 1) the entanglement subspaces of the ground state, for N_A particles out of N ; 2) spaces of amplitudes (conformal blocks) constructed from a CFT; and 3) spaces of zero-energy states of a special or projection operator Hamiltonian. In favorable cases, we can show that all of these consist of exactly the same wavefunctions. Even when they do not, it may be possible to add terms to the Hamiltonian and restore agreement, in terms of a “quotient” CFT. This is a finite-size version of a bulk-edge correspondence, in current parlance.

We have not addressed the issue of the energy gap above the zero-energy states for the special Hamiltonian. It is expected that when the CFT is non-unitary or non-rational, the model will be gapless in the thermodynamic limit [14]. We want to remark here that our analysis makes it clear that there are many (many-particle-interaction) terms that can be added to such a Hamiltonian, such that the zero-energy states are unchanged. The non-zero energy states would be affected, however. But it should be the case that no such terms can produce a gap in the bulk spectrum. This and the nature of the gapless systems remains a topic for future investigation.

ACKNOWLEDGMENTS

We thank E. Rezayi, N. Regnault, J. Dubail, and E. Ardonne for helpful discussions. This work was supported by NSF grants nos. DMR-0706195 (TSJ and NR) and DMR-1005895 (NR), and by EP-SRC grants nos. EP/I032487/1 and EP/I031014/1 (SHS).

Appendix A: Proof of Theorem 1

First, we prove that $a(z)$ and $a(z)^*$ generate the VOA. We take as given these operators at any locations z , and use translation invariance of ope’s; we must show that the current and Virasoro algebras are generated by these operators. To begin, we take the ope $a(z')a(z)$ as $z' \rightarrow z$

(z fixed), and extract the leading term, which contains the operator $e^{2i\varphi(z)/\sqrt{\nu}}$ times the leading operator in the $\psi\psi$ ope; the whole operator is located at any z . Then we fuse this with another a , and repeat the process. By definition of the properties of ψ , after taking the product of k a ’s, we obtain the operator $e^{ik\varphi(z)/\sqrt{\nu}}$ (times the identity in the ψ theory). Doing the same with a^* ’s produces $e^{-ik\varphi(z)/\sqrt{\nu}}$. The ope of these two operators in the charge sector (the φ CFT) contains the current operator $j(z)$, and the ope of two j s produces the stress tensor $T_\varphi(z)$. It follows that all descendants generated by currents (“current excitations”) of the leading operator in the ope $a(z')a(z)$ as $z' \rightarrow z$ are also in the algebra, and similarly for the leading operator when another $a(z'')$ approaches this, and so on.

Next, we consider again the ope of a with itself. In addition to the leading term referred to above, it also contains subleading terms, each of which is a sum of the leading and sub-leading terms in the $\psi\psi$ ope, possibly with current excitations, with weights that sum to the correct value corresponding to the overall power of z . The subleading terms consisting of current excitations only can be subtracted off, as they are already in the algebra. At each level (extra factor of z), that leaves an operator containing descendants in the algebra generated by ψ , times current excitations. At the first level, there is just one term, the first subleading part of the $\psi\psi$ ope (if it is non-zero), times $e^{2i\varphi(0)/\sqrt{\nu}}$, which is therefore also in the algebra. At the next level, the current descendants of this can be subtracted off as well, possibly leaving another operator with only descendants in the ψ algebra. Proceeding in this way, we find that all descendants in the $\psi(z')\psi(z)$ ope, times $e^{2i\varphi(0)/\sqrt{\nu}}$, times current excitations, are in the algebra. This can be repeated also in the ope of these operators with a further $a(z')$, and so on. Then we obtain recursively all of the VOA V , defined in Eq. (3.18). In particular, the stress tensor $T_\psi(z)$ is eventually generated.

We can now prove Theorem 1. We observe that the iterated ope of $k-1$ a ’s, taking the leading term in each ope, produces $e^{i(k-1)\varphi(z)/\sqrt{\nu}}\psi^*(z)$, by definition because a further ope with a gives $e^{ik\varphi(z)/\sqrt{\nu}}$. The ope with $e^{-ik\varphi(z)/\sqrt{\nu}}$ gives a^* as the leading term. Hence the VOA V is generated by $a(z)$ and $e^{-ik\varphi(z)/\sqrt{\nu}}$, which is the second statement in Theorem 1.

To obtain the first statement, we consider the operators $e^{iN\varphi(z)/\sqrt{\nu}}$, for N divisible by k , which (for $z=0$) correspond to the states $|0_N\rangle$. Acting on one of these with a ’s gives the space V^N , by definition. An ope with $e^{-ik\varphi(z)/\sqrt{\nu}}$ lowers N by k . Hence the union of the spaces of states (or corresponding operators) V^{-N} over all non-negative multiples of k , or the limit $\lim_{N \rightarrow \infty} V^{-N}$ using the embeddings, gives all of the VOA V . That is the first statement in Theorem 1.

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